Astro 507

Lecture 10

Feb. 12, 2020

Announcements:

- **Preflight 2 posted, due noon Friday**
  includes discussion question on the Anthropic Principle!
- Office Hours: Instructor after class today, or by appt
  TA: noon- 1pm tomorrow, or by appt

Last time:

- completed cosmic inventory: $\Omega_0 \approx 1$ and $\Omega_{\text{matter}} \approx 0.3$ Q: and so?
- high time to become relativistic
  introduced invariant interval Δs^2 *Q: wut? why?*
  particle with mass m, relativistic energy E has speed:
  $v(E) = \sqrt{1 - m^2/E^2}$ *Q: consequences?*

Causality and Spacetime

any particle of total energy E, mass m

moves at speed $v(E) = \sqrt{1 - \left(\frac{m}{E}\right)^2}$

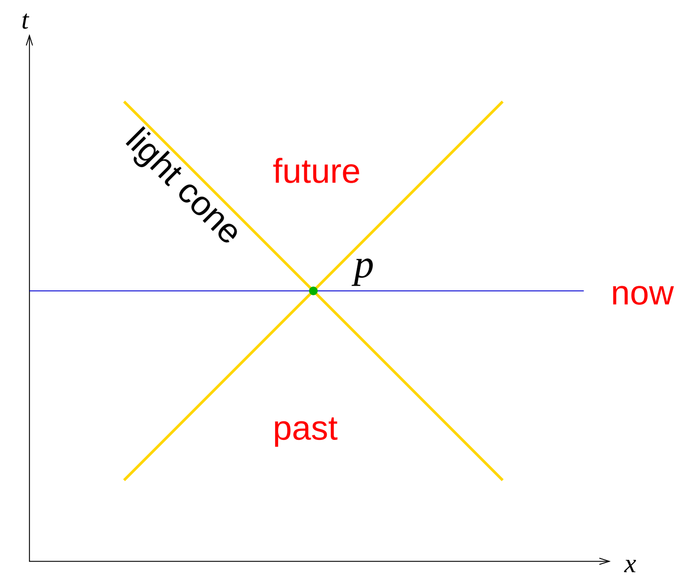
- massive particles have $0 \leq v < c$
- massless particles (e.g., γ) have $v = c$

$\Rightarrow$ *$v = c = 1$ is universal speed limit*

$\Rightarrow$ *cannot transmit particles, info any faster*

future light cone at spacetime point p
encloses region within which
particles/info can move
i.e., *region p can influence*
$\Rightarrow$ future light cone is spacetime region
causally connected to p

past light cone at p *Q: significance?*

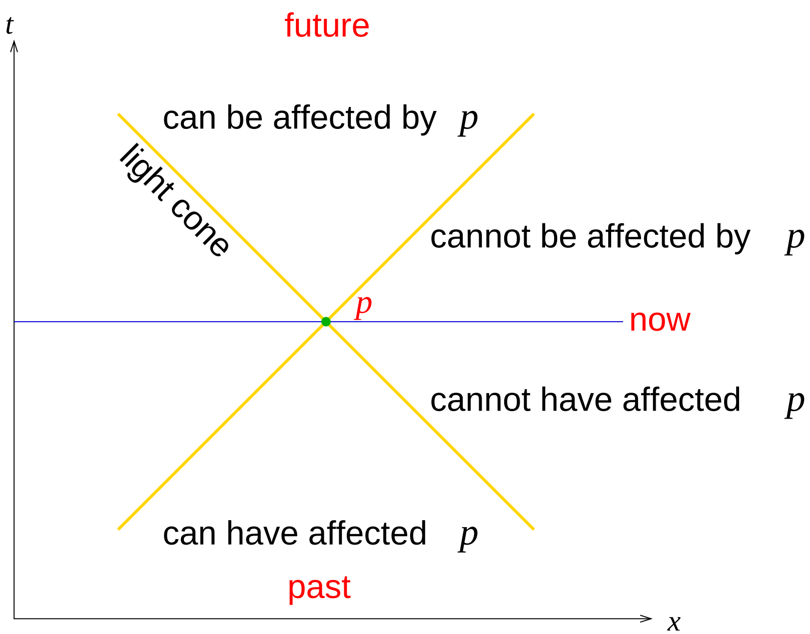


past light cone at p

events in cone can send particles/info to p

i.e., region which could have influenced p

$\Rightarrow$ past light cone=causally connected to p



Q: two events causally connected if?

Q: sufficient or just necessary?

What About Gravity?

A. Einstein (1905):

Newtonian dynamics $\rightarrow$ relativistic dynamics

space, time $\rightarrow$ spacetime forever more

Relativity and classical fields:

- E&M: Maxwell eqs relativistically *OK*! (*motivated* Lorentz, SR)
- Newtonian gravity: $\vec{g} = -\nabla\phi = -Gm/r^2\,\hat{r}$ and $\nabla^2\phi = 4\pi G\rho$
  an *unmitigated disaster* *Q: Why?*

How to fix?

First attempt: analogy with electrostatics *Q: why?*

$$\nabla^2\phi - \partial_t^2\phi = 4\pi G\rho \tag{1}$$

- bad news: disagrees with expt (gives no light bending)
- good news: right “flavor”
  e.g., operator $\nabla^2 - \partial_t^2 \rightarrow$ waves $\rightarrow$ gravitational radiation!

Mystic Pisa

Experiment: Galileo (Tower of Pisa?)
free fall independent of mass, size, shape, composition
Q: lawyer's fine print?

Theory: Newton
always: $\vec{F} = m\vec{a}$
gravity: mass is "coupling strength" $\Rightarrow \vec{F}_{\text{grav}} = m\vec{g}$
$\Rightarrow$ free fall has $\vec{a} = \vec{g} \rightarrow$ indep of object properties
interesting curiosity

Theory: Einstein
gravity is acceleration, so maybe *acceleration is gravity*
i.e., their physical effects indistinguishable/equivalent

Equivalence Principle

T-shirt summary (R. Wald):
all bodies fall the same way in a gravitational field

an observer in free fall *Q: meaning?*
cannot perform any experiment to determine whether
she is in a gravitational field

an observer undergoing acceleration
cannot perform any experiment to determine whether
she is in a gravity field or an accelerating spacecraft

Q: explain apple weight–Earth's surface
vs rocket accelerating $a = g$*?*
Q: explain apple drop–Earth's surface vs rocket with $a = g$*?*

Newton's Apple Experiment: Two Views

Isaac Newton on Earth's surface:

- holds an apple in his hand, **pushes up with force** $F = mg$
  says: must oppose weight so net force zero
- releases apple, **observes downward acceleration**
  says: motion due to net gravity force

Albert Einstein in rocket with constant acceleration $a = g$:

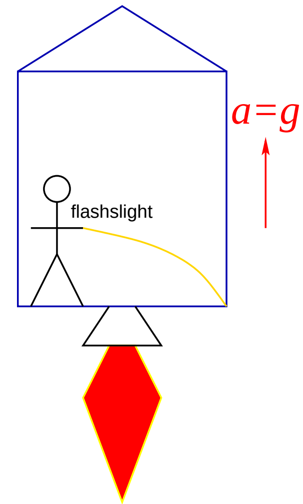
- holds apple in hand, **pushes up with force** $F = mg$
  says: to keep apple in my non-inertial accelerating hand must push so it accelerates too
- releases apple, **observes downward acceleration**
  says: motion due to my non-inertial frame

Note: identical physical results, radically different explanations

Q: what about horizontal ball toss?
Q: what about horizontal light beam?

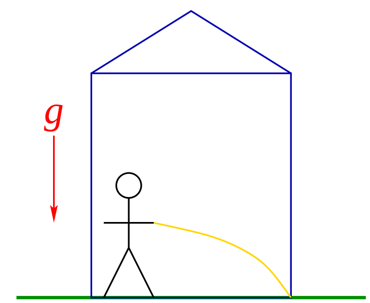
Gravity Bends Light

Rocket Experiment: `www: illuminating animation`
in accelerating rocket, shoot a horizontal beam
★ light ray deflected
★ entire light path bent (in fact, a parabola!)
"gravity's rainbow"



But by equivalence principle:
must find same result due to gravity, so:
★ gravity bends light rays

gravitational lensing



Acceleration and Photons

Still consider accelerating spaceship

Experiment: light signals between top & bottom
each flashes every $\tau_{\rm em} = 1$ sec
according to emitter's clock
emission frequency $t_{\rm em} = 1/\tau_{\rm em}$

Q: what is change in top clocks' speed when pulse arrives?
Q: what frequency does top clock observe?

asymmetry: top clock accelerates away from bottom flash
$\rightarrow$ relative speed changes during light transit
by amount $\delta v_{\rm top} = -a\delta t \simeq -ah/c$
sign $\rightarrow$ receding from source

$\rightarrow$ top observer sees freq Doppler shifted downward: *redshift*

$$f_{\rm obs,top} \approx \left(1 - \frac{\delta v}{c}\right) f_{\rm em,bottom} \tag{2}$$

so top observer sees bottom flash period as

$$\frac{\tau_{\rm obs} - \tau_{\rm em}}{\tau_{\rm em}} = \frac{\delta\tau}{\tau} = -\frac{\delta f}{f} \approx \frac{\delta v}{c} = +\frac{ah}{c^2} \tag{3}$$

Q: which means? and upon applying equivalence principle...?

Equivalence Principle and Photon Properties

Equivalence principle: gravitational results identical to rocket

- shifted frequencies: *gravitational redshift/blueshift!*
- period shift: *gravitational time dilation*

$$\frac{\delta t}{t} = \frac{\delta \lambda}{\lambda} \approx \frac{gh}{c^2} = \frac{\phi}{c^2} \tag{4}$$

attic clocks faster than basement clocks: verified experimentally!

`www: Pound-Rebka expt`

in weak gravity:

- fractional shift $\approx \phi/c^2$
- set by change in gravitational potential ϕ

Consequences of the Equivalence Principle

equivalence principle implies that gravity

- bends light trajectory: *distorts path in space*

- changes apparent frequency: *distorts apparent flow of time*

together these mean $\rightarrow$ **gravity alters spacetime!**

Einstein (1915): **include gravity in spacetime**

General Relativity

Newton (1687): Universal Gravitation
gravity is a force (field) that couples to mass
▷ matter tells gravity how to force
▷ gravity force tells matter how to move

Einstein (1915): General Relativity
gravity is spacetime curvature: not a force!
⋆ "matter tells spacetime how to curve
⋆ spacetime tells matter how to move" –J. .A. Wheeler

Curved Spacetime?

Curved space: geometric constructions in space
(triangles, rectangles, circles... *Q: how define?*)
give non-Euclidean results *Q: namely?*

Q: so–curved spacetime?

Spacetime Curvature

Test: (Feynman Lectures II, Chapter 42)

- construct geometric object in spacetime
- are properties Euclidean?

Case 1: Minkowski Space (i.e., special relativity, no accel)
(1-D) interval ("line element") for events separated by (dt, dx)

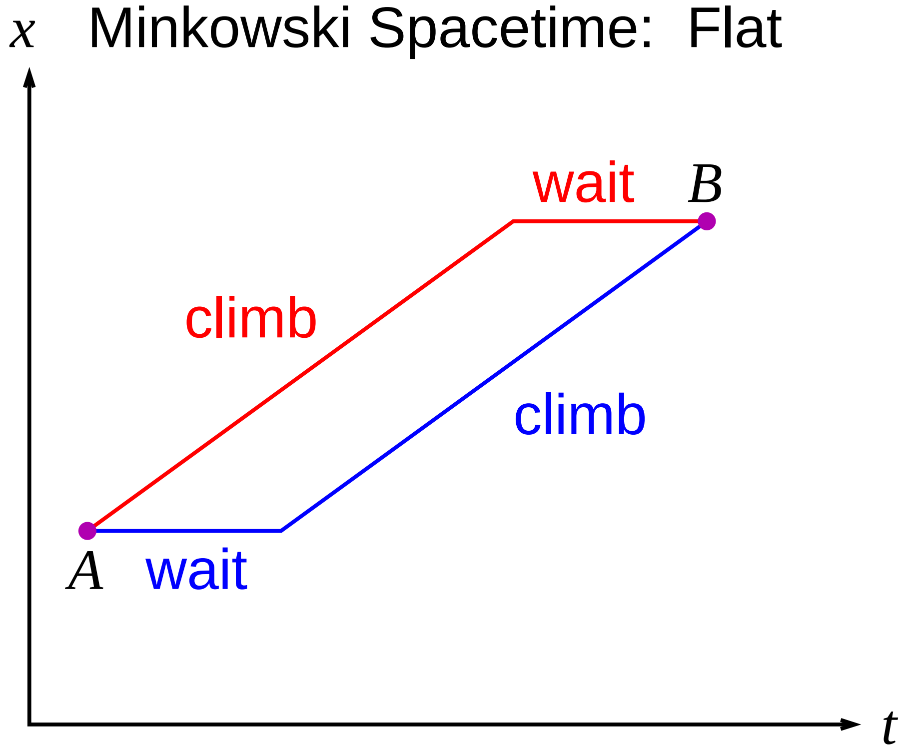
$$ds^2 = dt^2 - dx^2 \qquad (5)$$

Construct rhombus: in *spacetime*
two observers go from events A to B

▷ obs 1: go left at $v = 0.5c$ for 10 s, then wait 10 s
▷ obs 2: wait 10 s, then go left at $v = 0.5c$ for 10 s

Q: spacetime diagram?

Minkowski Spacetime: Flat



result **is Euclidean** *Q: why?*

$\Rightarrow$ Minkowski spacetime is **not curved = flat**

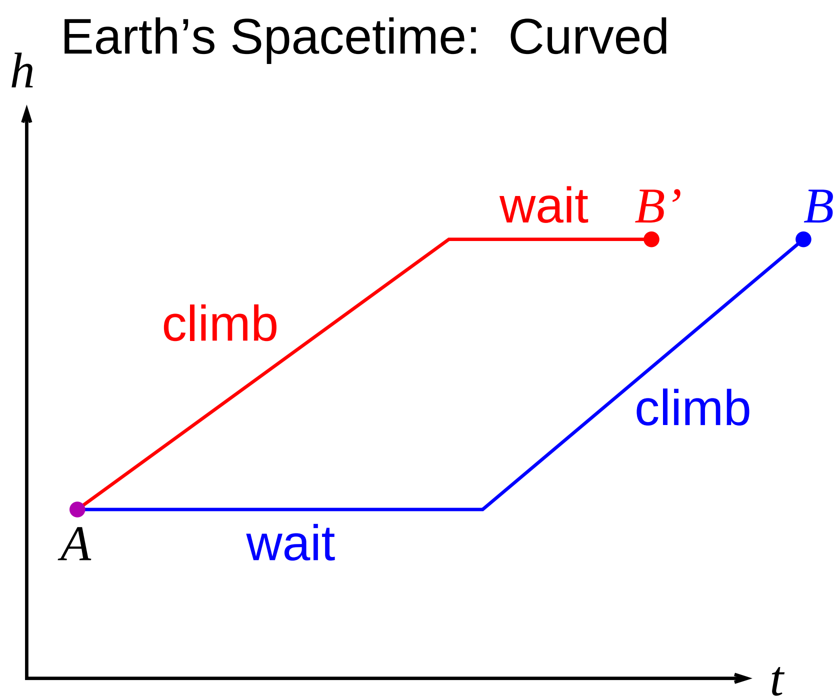
Case 2: Surface of Earth (i.e., const accel: gravity)
(1-D) line element:

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \left(1 + \frac{2\phi}{c^2}\right)^{-1} dx^2 \qquad (6)$$

where $\phi = \phi(x)$: time-independent Newtonian potential

Construct rhombus in *spacetime*
two observers go from events A to B
▷ obs 1: go up 1 km, then wait 10 s
▷ obs 2: wait 10 s, then go up 1 km

Q: spacetime diagram?



result is **not Euclidean**:

$$(\text{wait time}) = (\delta s)_{\text{wait}} = \sqrt{1 + 2gh/c^2}\ (\delta t)_{\text{wait}} \qquad (7)$$

why? waiting time "advance differently" – *time dilation*!

$\Rightarrow$ Earth's spacetime is **curved**!

gravity $\Leftrightarrow$ spacetime curvature

GR on a T-Shirt

General Relativity spirit and approach:
like special relativity, only moreso

Special Relativity concepts retained:

- **spacetime**: events, relationships among them
- **interval** gives observer-independent (invariant) measure of "distance" between events
- Special Relativity is a special case of GR
  SR: no gravity → no curvature → "flat spacetime"
  GR limit: gravity sources→0 give spacetime→Minkowski

GR: Special Relativity concepts generalized

- gravity encoded in spacetime structure
- spacetime can be curved
- coordinates have no intrinsic meaning

The Metric

Fundamental object in GR: **metric**

consider two nearby events, separated by

coordinate differences $dx = (dx^0, dx^1, dx^2, dx^3)$

GR (in orthogonal spacetimes) sez:

interval between them given by **"line element"**

$$\begin{aligned} ds^2 &= A(x)\,(dx^0)^2 - B(x)\,(dx^1)^2 - C(x)\,(dx^2)^2 - D(x)\,(dx^3)^2 \\ &\equiv \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \end{aligned}$$

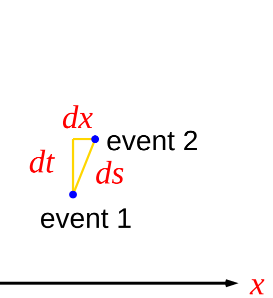
where the **metric tensor** $g_{\mu\nu}$

- in this case (orthogonal spacetime): $g = \text{diag}(A, B, C, D)$
- components generally are functions of space & *time* coords
- is symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$
- encodes all physics (like wavefunction in QM)

Q: if no gravity=Minkowski, what's the metric?

physical interpretation of interval: like in SR

$$ds^2 = (\text{apparent elapsed time})^2 - (\text{apparent spatial separation})^2$$



★ observers have *timelike* worldlines: $ds^2 > 0$

★ light has *null* $ds = 0$ worldlines

Simplest example: Minkowski space (Special Relativity)

$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$: constant values proper spatial distances:

- i.e., results using meter sticks
- measured simultaneously ($dx^0 = 0$)

length element:

$$d\ell^2 = -ds^2 = d\ell_1^2 + d\ell_2^2 + d\ell_3^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

space (3-)volume element:

$$\begin{aligned} dV_3 &= d\ell_1 d\ell_2 d\ell_3 \\ &= \sqrt{|g_{11}g_{22}g_{33}|}\; dx^1 dx^2 dx^3 \end{aligned}$$

spacetime 4-volume element:

$$\begin{aligned} dV_4 &= d\ell_0 dV_3 = \sqrt{|g_{00}g_{11}g_{22}g_{33}|}\; dx^0 dx^1 dx^2 dx^3 \\ &= \sqrt{|\det g|}\; dx^0 dx^1 dx^2 dx^3 \end{aligned}$$

Example: Minkowski space, Cartesian coords

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

length: $d\ell^2 = dx^2 + dy^2 + dz^2$

3-volume: $dV_3 = dx\,dy\,dz$

4-volume: $dV_4 = dx\,dy\,dz\,dt$

Example: Minkowski space, spherical coords

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

length: $d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

3-volume: $dV_3 = r^2 \sin\theta\, dr d\theta d\phi \equiv r^2 dr d\Omega$

4-volume: $dV_4 = r^2 dr d\Omega dt$

Relativistic Cosmology

Cosmological Spacetimes

Want to describe spacetime of the universe
to zeroth order: homogeneous, isotropic

1. at each spacetime point
   exactly *one* observer sees isotropy*
   call these fundamental observers
   roughly: "galaxies" i.e., us
   (strictly speaking, we don't qualify) *Q: why?*

2. isotropy at each point $\rightarrow$ homogeneity
   but can be homogeneous & not isotropic

*We will see: observers moving w.r.t. FOs eventually come to rest w.r.t. FOs

3. homogeneity and isotropy $\rightarrow$ symmetries
U. is **“maximally symmetric”**
$\rightarrow$ greatly constrain allowed spacetimes
i.e., allowed metrics

The Cosmic Line Element

cosmological principle:
can divide spacetime into time "slices"
i.e., 3-D spatial (hyper) surfaces

▷ populated by fundamental observers

▷ with coords, e.g., (t, x, y, z)

▷ choose FO's to have $d\vec{x} = 0$
i.e., spatial coords are **comoving** ("fixed to expanding grid")
on surface: fundamental observers must all have
$ds^2 = dt^2$ → i.e., $g_{tt} = const = 1$ *Q: why?*
→ g_{tt} *indep of space, time*

these give:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \quad (8)$$

Cosmological Principle and the Cosmic Metric

homogeneity and time

no space dependence on $d\ell_0 = dt$

- can define cosmic time t (FO clocks)
- at fixed t, time lapse dt not "warped" across space

homogeneity and space

- at any t, properties invariant under translations
- no center
- can pick arbitrary point to be origin
- e.g., here!

Cosmological spacetime encoded via cosmic **metric**
which determines how the interval depends on coordinates
any observer computes interval between events as
$ds^2 =$ (elapsed time)^2 – (spatial displacement)^2

Cosmic metric so far:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \tag{9}$$

where: t is cosmic time

now impose *isotropy*
- at any cosmic t, interval invariant under rotations
- pick arbitrary origin, then (comoving) spherical coords the usual r, θ, ϕ, with $r^2 = x^2 + y^2 + z^2$ and arbitrary origin (usually, but not always, here!)

Q: now that does metric look like?

For *fundamental* observers, maximal symmetry demands metric which can* be written as:

$$ds^2 = dt^2 - a(t)^2 d\ell^2_{\rm com} \tag{10}$$
$$= dt^2 - a(t)^2 \left[f(r) dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{11}$$

$a(t)$ is the cosmic scale factor

$f(r)$ is as yet undetermined

- for flat (Euclidean) space, $f(r) = 1$
- so $f \neq 1 \rightarrow$ non-Euclidean spatial geometry = curved space!

Q: why same time dep for radial and angular displacements?

Note power of cosmo principle

$\rightarrow$ only allowed dynamics is uniform expansion $a(t)$!

*other space & time coordinates possible and sometimes useful

but in all cases space and time must *factor* in this way