Astro 507

Lecture 11

Feb. 14, 2020

Announcements:

- **Preflight 2 was due at noon**
- **Problem Set 2 due next Friday**

- exciting cosmological Astronomy Colloquium Tue Feb 18
  Rachel Mandelbaum, Carnegie-Mellon
  "Cosmology with weak lensing
  in ongoing and upcoming imaging surveys"

GR on a T-Shirt

General Relativity spirit and approach:
like special relativity, only moreso

Special Relativity concepts retained:

- **spacetime**: events, relationships among them
- **interval** gives observer-independent (invariant) measure of "distance" between events
- Special Relativity is a special case of GR
  SR: no gravity → no curvature → "flat spacetime"
  GR limit: gravity sources→0 give spacetime→Minkowski

GR: Special Relativity concepts generalized

- gravity encoded in spacetime structure
- spacetime can be curved
- coordinates have no intrinsic meaning

The Metric

Fundamental object in GR: **metric**

consider two nearby events, separated by

coordinate differences $dx = (dx^0, dx^1, dx^2, dx^3)$

GR (in orthogonal spacetimes) sez:

interval between them given by **"line element"**

$$
\begin{aligned}
ds^2 &= A(x)\,(dx^0)^2 - B(x)\,(dx^1)^2 - C(x)\,(dx^2)^2 - D(x)\,(dx^3)^2 \\
&\equiv \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu
\end{aligned}
$$

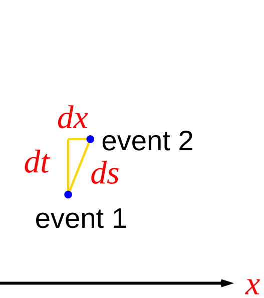
where the **metric tensor** $\boxed{g_{\mu\nu}}$

- in this case (orthogonal spacetime): $g = \mathrm{diag}(A, B, C, D)$
- components generally are functions of space & *time* coords
- is symmetric, i.e., $g_{\mu\nu} = g_{\nu\mu}$
- encodes all physics (like wavefunction in QM)

Q: if no gravity=Minkowski, what's the metric?

physical interpretation of interval: like in SR

$$ds^2 = (\text{apparent elapsed time})^2 - (\text{apparent spatial separation})^2$$



⋆ observers have *timelike* worldlines: $ds^2 > 0$

⋆ light has *null* $ds = 0$ worldlines

Simplest example: Minkowski space (Special Relativity)

$g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$: constant values

Relativistic Cosmology

Cosmological Spacetimes

Want to describe spacetime of the universe
to zeroth order: homogeneous, isotropic

1. at each spacetime point
   exactly *one* observer sees isotropy*
   call these **fundamental observers**
   roughly: "galaxies" i.e., us
   (strictly speaking, we don't qualify) *Q: why?*

2. isotropy at each point $\rightarrow$ homogeneity
   but can be homogeneous & not isotropic

*We will see: observers moving w.r.t. FOs eventually come to rest w.r.t. FOs

3. homogeneity and isotropy $\rightarrow$ symmetries

U. is **“maximally symmetric”**

$\rightarrow$ greatly constrain allowed spacetimes

i.e., allowed metrics

Cosmological Principle and Cosmic Spacetime

Executive Summary

Cosmo Principle $\rightarrow$ at any time, space is **maximally symmetric**

- strongly restricts allowed spacetime structure
- there exist a set of **fundamental observers** (FOs)
  (or "frames" or "coordinate systems")
  who see U as homogenous and isotropic
- FOs "ride on" or are at rest w.r.t. **comoving coordinates**
  which don't change with expansion
  but do of course physically move apart
- FO clocks all tick at same rate, measure **cosmic time** t

Note: in a generic spacetime, not possible to "synchronize clocks" in this way

Spaces of Constant Curvature

Amazing mathematical result:
despite enormous constraints of maximal symmetry
GR does *not* demand cosmic space to be flat (Euclidean)
as assumed in pre-relativity and special relativity

GR allows *three classes of cosmic spatial geometry*
each of which is a space of constant (or zero) curvature

- positive curvature → hyper-spherical
- negative curvature → hyperbolic
- zero curvature → flat (Euclidean)

www: cartoons

All of these are *allowed* by GR and maximal symmetry
but *our* universe can have only *one* of them
Q: how do we know which of these our U has "chosen"?

Positive Curvature: A (Hyper-)Spherical Universe

to get an intuition: consider ordinary sphere ("2-sphere")
using coordinates in Euclidean space ("embedding")
sphere defined by

$$(x, y, z) \ \in \ x^2 + y^2 + z^2 = R^2 = const \tag{1}$$

Coordinates on the sphere:

- *usual* spherical coords: center, *origin outside of the space*
- we will use coordinates with origin in the space
  more convenient, closer to the physics *Q: why?*

origin: **at north pole**

$(x, y, z) = (0, 0, +R)$

r distance from z-axis

$r \Leftrightarrow$ latitudes

$r^2 = x^2 + y^2 = R^2 - z^2$

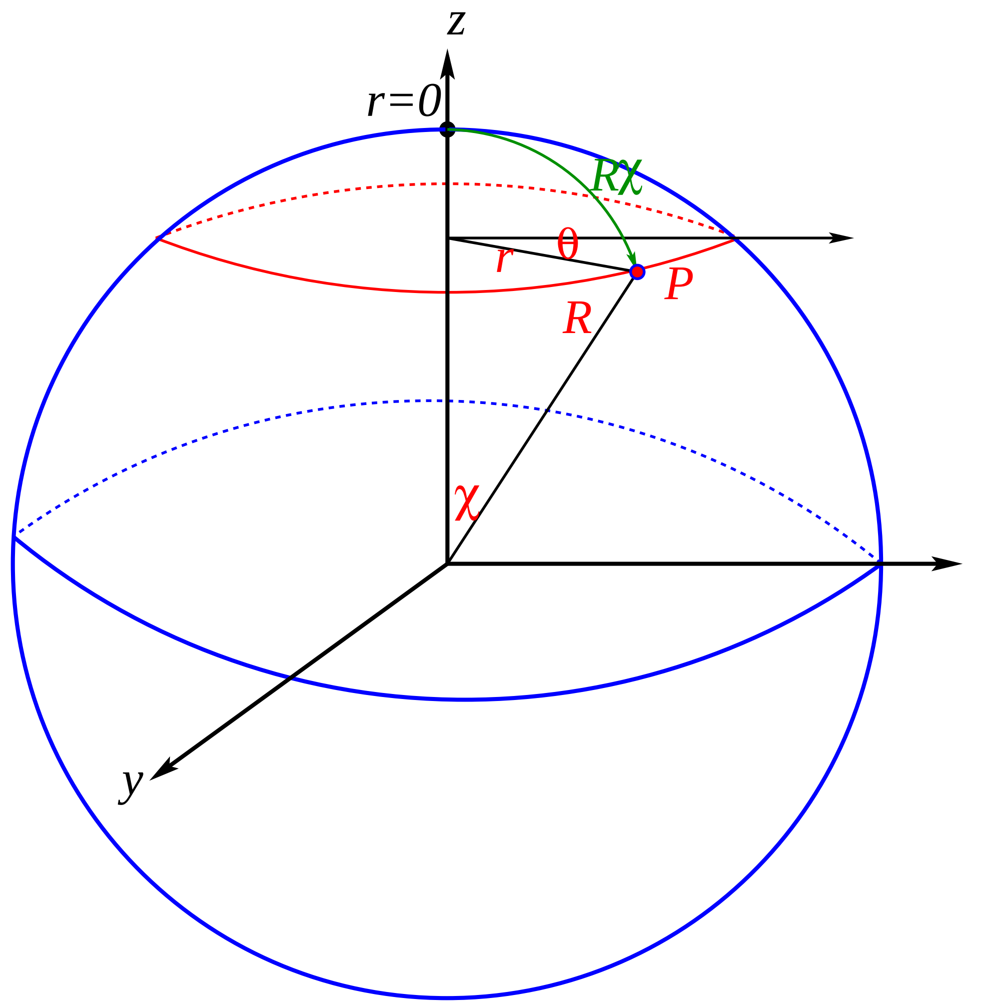
θ angle from x axis

$\theta \rightarrow$ longitude

$R\,\chi$

arclength on sphere from pole

χ is usual spherical polar angle



2-sphere metric:

in 3-D embedding space: $d\ell^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$

but points, intervals constrained to lie on sphere:

$R^2 = r^2 + z^2 = const$

$d(R^2) = 0 = xdx + ydy + zdz = rdr + zdz$

so $dz = -rdr/z \rightarrow$ can eliminate z

thus in polar coords with origin at N Pole

$$d\ell^2 = dr^2 + r^2 d\theta^2 + dz^2 = \left(1 + \frac{r^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2 \quad (2)$$

$$= \left(\frac{R^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 \quad (3)$$

not the Euclidean expression!

curved space: curvature $R^2 = const$!

Exploring Sphereland

coordinates for (2-D) observers on sphere, centered at N Pole:

$$d\ell^2 = d\ell_r^2 + d\ell_\theta^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 = R^2 d\chi^2 + R^2 \sin^2\chi \, d\theta^2$$

N Pole inhabitant (2-Santa) measures radial distance from home:
$d\ell_r = dr/\sqrt{1 - r^2/R^2} \equiv R d\chi$
$\rightarrow$ radius is $\ell_r = R \sin^{-1}(r/R) \equiv R\chi$

Example: construct a *circle*
locus of points at same radius ℓ_r

- circumference $dC = d\ell_\theta = r d\theta = R \sin\chi d\theta$
  $\rightarrow C = 2\pi R \sin\chi < 2\pi \ell_r$
- area $dA = d\ell_r d\ell_\theta = R^2 \sin\chi \, d\chi d\theta$
  $\rightarrow A = 2\pi R^2 (1 - \cos\chi) < \pi \ell_r^2$

Q: why are these right?

3-D Life in a 4-D Sphere

generalize to 3-D “surface” of sphere in 4-D space (“3-sphere”), constant positive curvature R:
3-D spherical coordinates centered on “N pole”

spatial line element

$$d\ell^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{4}$$

- sky still has solid angle $d\Omega = \sin\theta \, d\theta \, d\pi$, $\int d\Omega = 4\pi$
- radial (proper) distance $\Delta\ell_r = R \sin^{-1}(r/R) \equiv R\chi$
- so we have found, for $\kappa = +1$,
  RW metric has $f(r) = 1/(1 - r^2/R^2)$

Q: guesses for zero, negative curvature metrics?

Friedmann-Lemaître-Robertson-Walker Metric

Robertson & Walker:
maximal symmetry imposes metric form

Robertson-Walker line element (in my favorite units, coords):

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

where cosmic geometry encoded via κ:

$$\kappa = \begin{cases} +1 & \text{pos curv: ``spherical''} \\ 0 & \text{flat: ``Euclidean''} \\ -1 & \text{neg curv: ``hyperbolic''} \end{cases} \tag{5}$$

Friedmann-Lemaître-Robertson-Walker Metric

Robertson & Walker:
maximal symmetry imposes metric form

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right)$$

where cosmic geometry encoded via κ:

$$\kappa = \begin{cases} +1 & \text{pos curv: ``spherical''} \\ 0 & \text{flat: ``Euclidean''} \\ -1 & \text{neg curv: ``hyperbolic''} \end{cases} \quad (6)$$

gives **interval** for neighboring events

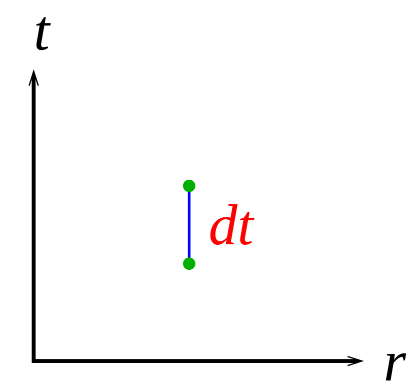
Consider event pairs (t, r, θ, ϕ) and $(t + \delta t, r, \theta, \phi)$

- *Q: what is* ds^2*?*
- *Q: what does* ds^2 *tell us physically?*

$$
\begin{aligned}
ds^2 &= dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) \\
&= (\text{apparent elapsed time})^2 - (\text{apparent distance})^2
\end{aligned}
$$

event separation $(dt, dr, d\theta, d\phi) = (\delta t, 0, 0, 0)$

- spatial coords unchanged:
  events at rest w.r.t. FO frame
- FO's apparent elapsed time is



$$ds = \delta t$$

lesson: dt is FO clock rate = **cosmic time**

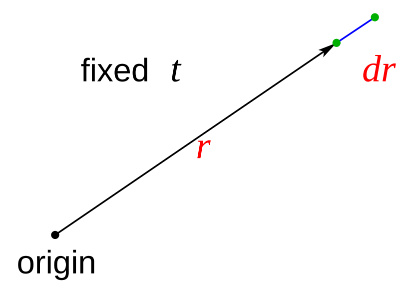
now consider pair: (t, r, θ, ϕ) and $(t, r + \delta r, \theta, \phi)$

Q: what is ds^2? physical significance?

$$
\begin{aligned}
ds^2 &= dt^2 - a(t)^2 \left(\frac{dr^2}{1-\kappa r^2/R^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right) \\
&= (\text{apparent elapsed time})^2 - (\text{apparent distance})^2
\end{aligned}
$$

event separation $(dt, dr, d\theta, d\phi) = (0, \delta r, 0, 0)$

- time coords unchanged:
  events simultaneous in FO frame
  $\Rightarrow ds^2$ gives $-(\text{apparent distance})^2 = -d\ell^2$
- separation is radial only
  $\Rightarrow$ FO finds physical radial distance is



$$
d\ell = d\ell_r = a(t)\, \frac{\delta r}{\sqrt{1-\kappa r^2/R^2}}
$$

Q: lessons?

for event separation $(dt, dr, d\theta, d\phi) = (0, \delta r, 0, 0)$
physical radial distance is

$$d\ell = d\ell_r = a(t)\ \frac{\delta r}{\sqrt{1 - \kappa r^2/R^2}} \tag{7}$$

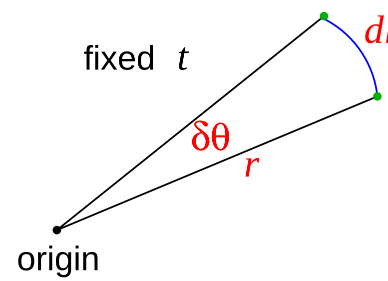
lessons:

- radial distances sensitive to curvature R
  not directly measured by r unless $\kappa = 0$
- radial distances evolve as $a(t)$ – of course!
- cosmoving radial distance is $d\ell_{r,\mathrm{com}} = \delta r/\sqrt{1 - \kappa r^2/R^2}$

now consider pair (t, r, θ, ϕ) and $(t, r, \theta + \delta\theta, \phi)$
Q: what is ds^2? physical significance?

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right)$$
$$= \text{(apparent elapsed time)}^2 - \text{(apparent distance)}^2$$

event separation $(dt, dr, d\theta, d\phi) = (0, 0, \delta\theta, 0)$



- time coords unchanged: events give FO distance
- separation is angular only
  $\Rightarrow$ FO finds distance = *arc length*

$$d\ell = d\ell_\theta = a(t)\ r\ \delta\theta \tag{8}$$

- arc lengths depend on radial coord r
  $\neq$ physical radial distance unless $\kappa = 0$
- arc lengths evolve as $a(t)$ – of course!
- comoving angular distance is $d\ell_{\theta,\mathrm{com}} = r\ \delta\theta$
- similarly, $d\ell_\phi = a(t)\ r\ \sin(\theta)\ \delta\phi$

consider a region with

- $dt = dr = 0$, and
- $d\theta, d\phi \neq 0$

Q: physical significance?

Q: relevant quantity?

consider a region with

- $dt = 0$
- $dr, d\theta, d\phi \neq 0$

Q: physical significance?

Q: relevant quantity?

region with $dt = dr = 0$ and $d\theta, d\phi \neq 0$:

- fixed time coordinate: events give spatial separation
- fixed radial coordinate r: separation is angular only
- both angular coordinates vary: sweeps 2-D region on sphere
- *area* of region is

$$dA = d\ell_\theta \; d\ell_\phi = a(t)^2 \; r^2 \; \sin(\theta) \; d\theta \; d\phi = a(t)^2 \; r^2 \; d\Omega \tag{9}$$

lesson:

- *solid angle* is usual $d\Omega = dA/a(t)^2 r^2 = \sin(\theta) \; d\theta \; d\phi$
- physical area of sphere at r is $A_{\text{sph}} = 4\pi \; a(t)^2 \; r^2$

region with $dt = 0$ and $dr, d\theta, d\phi \neq 0$:

- sweep out 3-D *spatial volume* on sphere

$$dV = d\ell_r \; d\ell_\theta \; d\ell_\phi = a(t)^3 \; \frac{r^2}{\sqrt{1 - \kappa r^2/R^2}} \; dr \; d\Omega \tag{10}$$

Friedmann-Lemaître-Robertson-Walker Cosmology

Friedmann & Lemaître:
solve GR dynamics (Einstein equation)
for stress-energy of "perfect fluid" (no dissipation)

The Einstein Equation and Robertson-Walker

Einstein eq: $R_{\mu\nu} - 1/2\, R g_{\mu\nu} = 8\pi G T_{\mu\nu}$
derivatives in Einstein eq come from curvature tensor $R_{\mu\nu}$
$\rightarrow$ schematically: "$R \sim \partial^2 g \sim G\rho$" – like Newtonian Poisson eq
but the only undetermined function in the metric
is the scale factor a, which only depends on t:
so: Einstein eqs $\rightarrow$ ODEs which set evolution of $a(t)$
$\Rightarrow$ these are the Friedmann equations!
and: in RW metric, local energy conservation $\nabla_\nu T^{\mu\nu} = 0$
$\Rightarrow$ gives 1st Law: $d(\rho a^3) = -p d(a)^3$

More detail in today's Director's Cut Extras

Life in a FRLW Universe

FLRW metric + Friedmann eqs for $a(t)$
$\rightarrow$ all you need to calculate anything
particle motions, fluid evolution, observables...

Excellent first example: Propagation of light

We want to know

- photon path through spacetime
- evolution of photon λ, E during propagation
- detected redshift

Q: how to calculate these?
Q: relevant equations?
Q: coordinate choices?

Worked Example: Photon Propagation

photon path: radial null trajectory $ds = 0$ (Fermat)

★ emitted at $r_{\rm em}$, $t_{\rm em}$

★ observed at $r_{\rm obs} = 0$, $t_{\rm obs}$

for FOs at $r_{\rm em}$ and $r_{\rm obs} = 0$,
any $t_{\rm em}$ and $t_{\rm obs}$ pairs have

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} = \int_0^{r_{\rm em}} \frac{dr}{\sqrt{1-\kappa r^2/R^2}}$$

time-dep   time-indep

Since RHS is time-independent *Q: why?*
then *any* two pairs of emission/observation events
between comoving points $r \to 0$ must have

$$\int_{t_{\rm em,1}}^{t_{\rm obs,1}} \frac{dt}{a(t)} = \int_{t_{\rm em,2}}^{t_{\rm obs,2}} \frac{dt}{a(t)} \tag{11}$$

consider two sequential emission events, lagged by δt_{em}
subsequently seen as sequential observation events with δt_{obs}

time-independence of propagation integral means

$$\int_{t_{\text{em}}}^{t_{\text{obs}}} \frac{dt}{a(t)} = \int_{t_{\text{em}}+\delta t_{\text{em}}}^{t_{\text{obs}}+\delta t_{\text{obs}}} \frac{dt}{a(t)}$$

rearranging...

$$\int_{t_{\text{em}}}^{t_{\text{em}}+\delta t_{\text{em}}} \frac{dt}{a(t)} = \int_{t_{\text{obs}}}^{t_{\text{obs}}+\delta t_{\text{obs}}} \frac{dt}{a(t)}$$

if δt small (*Q: compared to what?*)
then $\delta t_{\text{em}}/a(t_{\text{em}}) = \delta t_{\text{obs}}/a(t_{\text{obs}})$ and so

$$\frac{\delta t_{\text{obs}}}{\delta t_{\text{em}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})}$$

Q: observational implications?

Observational implications:

⋆ for *any* pairs of photons

$$\frac{\delta t_{\rm obs}}{\delta t_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} = \frac{1+z_{\rm em}}{1+z_{\rm obs}}$$

and since $a(t_{\rm obs}) > a(t_{\rm em})$

$\rightarrow$ $\delta t_{\rm obs} > \delta t_{\rm em}$

$\rightarrow$ distant happenings appear in slow motion!

$\rightarrow$ time dilation!

cosmic time dilation recently observed!

Q: how would effect show up?

Q: why non-trivial to observationally confirm?

 `www: cosmic time dilation evidence`

Director's Cut Extras

proper spatial distances:

- i.e., results using meter sticks
- measured simultaneously ($dx^0 = 0$)

length element:

$$d\ell^2 = -ds^2 = d\ell_1^2 + d\ell_2^2 + d\ell_3^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

space (3-)volume element:

$$\begin{aligned} dV_3 &= d\ell_1 d\ell_2 d\ell_3 \\ &= \sqrt{|g_{11}g_{22}g_{33}|}\, dx^1 dx^2 dx^3 \end{aligned}$$

spacetime 4-volume element:

$$\begin{aligned} dV_4 &= d\ell_0 dV_3 = \sqrt{|g_{00}g_{11}g_{22}g_{33}|}\, dx^0 dx^1 dx^2 dx^3 \\ &= \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3 \end{aligned}$$

Example: Minkowski space, Cartesian coords

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

length: $d\ell^2 = dx^2 + dy^2 + dz^2$

3-volume: $dV_3 = dx\,dy\,dz$

4-volume: $dV_4 = dx\,dy\,dz\,dt$

Example: Minkowski space, spherical coords

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

length: $d\ell^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

3-volume: $dV_3 = r^2 \sin\theta\, dr d\theta d\phi \equiv r^2 dr d\Omega$

4-volume: $dV_4 = r^2 dr d\Omega dt$

The Cosmic Line Element

cosmological principle:
can divide spacetime into time "slices"
i.e., 3-D spatial (hyper) surfaces

▷ populated by fundamental observers

▷ with coords, e.g., (t, x, y, z)

▷ choose FO's to have $d\vec{x} = 0$
i.e., spatial coords are **comoving** ("fixed to expanding grid")
on surface: fundamental observers must all have
$ds^2 = dt^2 \to$ i.e., $g_{tt} = const = 1$ *Q: why?*
$\to g_{tt}$ *indep of space, time*

these give:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \tag{12}$$

Cosmological Principle and the Cosmic Metric

homogeneity and time

no space dependence on $d\ell_0 = dt$

- can define cosmic time t (FO clocks)
- at fixed t, time lapse dt not "warped" across space

homogeneity and space

- at any t, properties invariant under translations
- no center
- can pick arbitrary point to be origin
- e.g., here!

Cosmological spacetime encoded via cosmic **metric**
which determines how the interval depends on coordinates
any observer computes interval between events as
$ds^2 =$ (elapsed time)^2 – (spatial displacement)^2

Cosmic metric so far:

$$ds^2 = dt^2 - g_{ii}(dx^i)^2 \tag{13}$$

where: t is cosmic time

now impose *isotropy*

- at any cosmic t, interval invariant under rotations
- pick arbitrary origin, then (comoving) spherical coords
  the usual r, θ, ϕ, with $r^2 = x^2 + y^2 + z^2$
  and arbitrary origin (usually, but not always, here!)

Q: now that does metric look like?

For *fundamental* observers, maximal symmetry demands metric which can* be written as:

$$ds^2 = dt^2 - a(t)^2 d\ell^2_{\rm com} \quad (14)$$

$$= dt^2 - a(t)^2 \left[f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (15)$$

$a(t)$ is the cosmic scale factor

$f(r)$ is as yet undetermined

- for flat (Euclidean) space, $f(r) = 1$
- so $f \neq 1 \rightarrow$ non-Euclidean spatial geometry = curved space!

Q: why same time dep for radial and angular displacements?

Note power of cosmo principle

$\rightarrow$ only allowed dynamics is uniform expansion $a(t)$!

*other space & time coordinates possible and sometimes useful

but in all cases space and time must *factor* in this way

Curvature

maximal symmetry requires that Universe spatial "3-volume" is a "**space of constant curvature**"

at any time t: cosmic curvature is a length $\mathcal{R}(t)$
- today: $\mathcal{R}(t_0) \equiv R$
- *Q: dependence on scale factor?*

For the relativists: max symmetry means *spatial* curvature tensor must take the form

$$R^{(3)}_{ijk\ell} = \frac{\kappa}{\mathcal{R}(t)^2} \left(h_{ik} h_{jl} - h_{jk} h_{il} \right) \tag{16}$$

where $\kappa = -1$, 0, or $+1$
and h is the spatial part of metric g

Note: the curvature scalar is really one single number K

but for $K \neq 0$ one can identify a sign $\kappa \equiv K/\|K\|$ and lengthscale $\mathcal{R}^2 \equiv 1/\|K\|$

Perfect fluid:

- "perfect" $\rightarrow$ no dissipation (i.e., viscosity)
- stress-energy: given density, pressure fields ρ, p and 4-velocity field $u_\mu \rightarrow (1,0,0,0)$ for FO

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(g_{\mu\nu} - u_\mu u_\nu) \tag{17}$$

$$= \mathrm{diag}(\rho, p, p, p)_{\mathrm{FO}} \tag{18}$$

Recall: stress-energy conservation is

$$\nabla_\nu T^{\mu\nu} = 0 \tag{19}$$

where ∇_μ is covariant derivative

For RW metric, this becomes:

$$d(a^3 \rho) = p d(a^3) \tag{20}$$

1st Law of Thermodynamics!

Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{21}$$

Given RW metric (orthogonal, max symmetric):

- *Q: how many nonzero Einstein eqs generally? here?*
- *Q: what goes into $G_{\mu\nu}$? what will this be for RW metric?*

Einstein eq:

$G_{\mu\nu}, T_{\mu\nu}$ symmetric 4×4 matrices $\rightarrow$ 10 independent components

in general, Einstein $\rightarrow$ 10 equations

but cosmo principle demands: space-time terms $G_{0i} = 0$

and off-diagonal space-space $G_{ij} = 0$

else pick out special direction $\Rightarrow$ only diagonal terms nonzero

and all 3 "p" equations same

Einstein $\rightarrow$ two independent equations

$$
\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} && (22) \\
&= 8\pi G T_{00} = 8\pi G\rho && (23) \\
G_{ii} &= 6\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} && (24) \\
&= 8\pi G T_{ii} = 8\pi G p && (25)
\end{aligned}
$$

After rearrangement, these become
the Friedmann "energy" and acceleration equations!