Astro 507

Lecture 12

Feb. 17, 2020

Announcements:

- **Problem Set 2 due Friday**
- **No Class Meeting this Wed and Fri, Feb 19 and 21**
  time off for good behavior, instructor travel
  Instructor available via Homework Discussion page on Compass
  TA Office Hours noon-1pm Thursday

- exciting cosmological Astronomy Colloquium Tue Feb 18
  Rachel Mandelbaum, Carnegie-Mellon
  "Cosmology with weak lensing
  in ongoing and upcoming imaging surveys"

Last time: Robertson-Walker metric

Q: what is it?

Q: parameters? variables?

Q: what coordinate system?

Q: what does it mean physically?

Friedmann-Lemaître-Robertson-Walker Metric

Robertson & Walker:
maximal symmetry imposes metric form

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2/R^2} + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

- variables s, t, r, θ, ϕ
- parameters: R gives comoving curvature length, and cosmic geometry encoded via κ:

$$\kappa = \begin{cases} +1 & \text{pos curv: “spherical”} \\ 0 & \text{flat: “Euclidean”} \\ -1 & \text{neg curv: “hyperbolic”} \end{cases} \tag{1}$$

ω

metric gives **interval** for neighboring events
Q: interval and meaning for (t, r, θ, ϕ) $(t, r + \delta r, \theta, \phi)$*?*

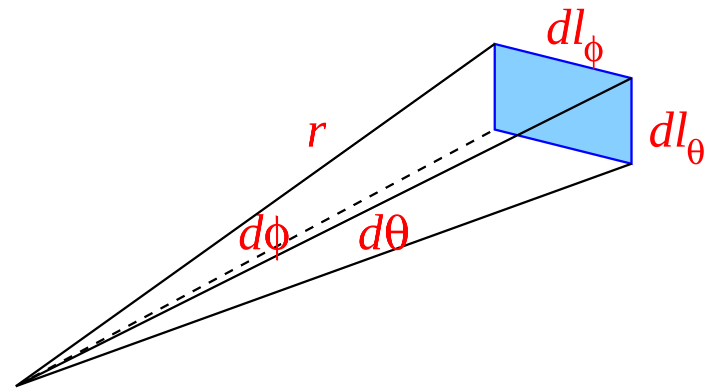
Exploring the Robertson-Walker Metric

consider a spacetime region with

- $dt = dr = 0$, and
- θ and ϕ independently sweep $d\theta, d\phi \neq 0$

Q: physical significance?

Q: relevant quantity?



a spacetime region with $dt = dr = 0$
and $d\theta, d\phi \neq 0$:

- fixed time coordinate: events give spatial separation
- fixed radial coordinate r: separation is angular only
- both angular coordinates vary: sweeps 2-D region on sphere
- *physical area* of region is

$$dA = d\ell_\theta \ d\ell_\phi = a(t)^2 \ r^2 \ \sin(\theta) \ d\theta \ d\phi = a(t)^2 \ r^2 \ d\Omega \qquad (2)$$

lesson:

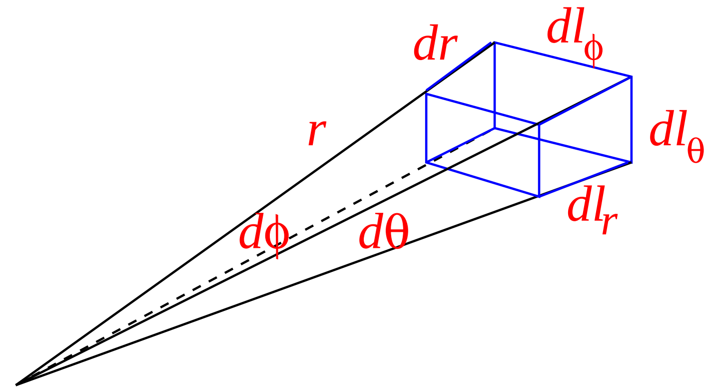
- physical **area of sphere with radius** r is $A_{\rm sph} = 4\pi \ a(t)^2 \ r^2$
  note $A \propto a^2$ scaling appropriate for a physical area
- *solid angle* is $d\Omega = dA/A_{\rm sph} = \sin(\theta) \ d\theta \ d\phi$ as usual!

consider a region with

- $dt = 0$
- (r, θ, ϕ) independently sweep $dr, d\theta, d\phi \neq 0$

Q: physical significance? relevant quantity?

for spacetime region with $dt = 0$
and $dr, d\theta, d\phi \neq 0$ all vary independently:



- sweep out 3-D *spatial volume* on sphere

$$dV = d\ell_r \; d\ell_\theta \; d\ell_\phi \tag{3}$$

$$= a(t)^3 \frac{r^2}{\sqrt{1 - \kappa r^2/R^2}} \, dr \; \sin(\theta) \; d\theta \; d\phi \tag{4}$$

$$= a(t)^3 \frac{r^2}{\sqrt{1 - \kappa r^2/R^2}} \, dr \; d\Omega \tag{5}$$

- physical volume scales as $dV \propto a^3$: check!

- for $\kappa \neq 0$ sphere volume not just r^3!

Friedmann-Lemaître-Robertson-Walker Cosmology

Friedmann & Lemaître:
solve GR dynamics (Einstein equation)
for stress-energy of "perfect fluid" (no dissipation)

The Einstein Equation and Robertson-Walker

Einstein eq: $R_{\mu\nu} - 1/2\, R g_{\mu\nu} = 8\pi G T_{\mu\nu}$
derivatives in Einstein eq come from curvature tensor $R_{\mu\nu}$
$\rightarrow$ schematically: "$R \sim \partial^2 g \sim G\rho$" – like Newtonian Poisson eq
but the only undetermined function in the metric
is the scale factor a, which only depends on t:
so: Einstein eqs $\rightarrow$ ODEs which set evolution of $a(t)$
$\Rightarrow$ these are the Friedmann equations!
and: in RW metric, local energy conservation $\nabla_\nu T^{\mu\nu} = 0$
∞ $\Rightarrow$ gives 1st Law: $d(\rho a^3) = -p d(a)^3$

More detail in today's Director's Cut Extras

Life in a FRLW Universe

FLRW metric + Friedmann eqs for $a(t)$
$\rightarrow$ all you need to calculate anything
particle motions, fluid evolution, observables...

Excellent first example: propagation of light

We want to know
- *photon path through spacetime*
- evolution of photon λ, E during propagation
- detected redshift

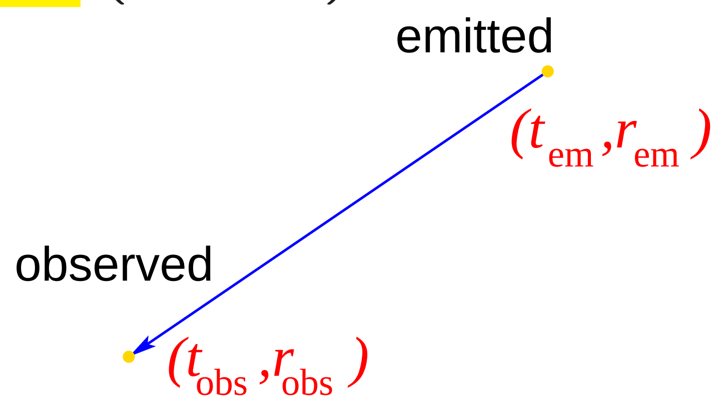
Q: how to calculate these?
Q: relevant equations?
Q: coordinate choices?

Worked Example: Photon Propagation

photon path: radial *null trajectory* $ds = 0$ (Fermat)

★ emitted at $r_{\rm em}$, $t_{\rm em}$

★ observed at $r_{\rm obs} = 0$, $t_{\rm obs}$



for FOs at $r_{\rm em}$ and $r_{\rm obs} = 0$,
any $t_{\rm em}$ and $t_{\rm obs}$ pairs have

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} = \int_0^{r_{\rm em}} \frac{dr}{\sqrt{1 - \kappa r^2/R^2}}$$

time-dep   time-indep

Since RHS is time-independent *Q: why?*
then *any* two pairs of emission/observation events
between comoving points $r \to 0$ must have

$$\int_{t_{\rm em,1}}^{t_{\rm obs,1}} \frac{dt}{a(t)} = \int_{t_{\rm em,2}}^{t_{\rm obs,2}} \frac{dt}{a(t)} \quad (6)$$

consider two sequential emission events, lagged by $\delta t_{\rm em}$
subsequently seen as sequential observation events with $\delta t_{\rm obs}$

time-independence of propagation integral means

$$\int_{t_{\rm em}}^{t_{\rm obs}} \frac{dt}{a(t)} = \int_{t_{\rm em}+\delta t_{\rm em}}^{t_{\rm obs}+\delta t_{\rm obs}} \frac{dt}{a(t)}$$

rearranging...

$$\int_{t_{\rm em}}^{t_{\rm em}+\delta t_{\rm em}} \frac{dt}{a(t)} = \int_{t_{\rm obs}}^{t_{\rm obs}+\delta t_{\rm obs}} \frac{dt}{a(t)}$$

if δt small (*Q: compared to what?*)
then $\delta t_{\rm em}/a(t_{\rm em}) = \delta t_{\rm obs}/a(t_{\rm obs})$ and so

$$\frac{\delta t_{\rm obs}}{\delta t_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})}$$

Q: observational implications?

Observational implications:

⋆ for *any* pairs of photons

$$\frac{\delta t_{\mathrm{obs}}}{\delta t_{\mathrm{em}}} = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})}$$

observed pulse interval differs from emitted duration due to scale factor change

- Consider monochromatic photons with rest wavelength λ_{em}
  Q: what if duration $\delta t_{\mathrm{em}} = \lambda_{\mathrm{em}}/c$*?*

Implications of Photon Propagation: Redshift Revisited

for monochromatic emission, $\delta t_{\mathrm{em}} = \lambda_{\mathrm{em}}/c = 1/f_{\mathrm{em}}$
is the time between wave crests, i.e., the wave period
which changes as

$$\frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}} = \frac{a(t_{\mathrm{obs}})}{a(t_{\mathrm{em}})}$$

- *wavelengths grow with scale factor!*
- verifies the "wavelengths are lengths" heuristic argument
- and using the definition of redshift, we again have

$$\frac{a_{\mathrm{obs}}}{a_{\mathrm{em}}} = \frac{1 + z_{\mathrm{em}}}{1 + z_{\mathrm{obs}}}$$

Note: one-to-one relationships
redshift $z \leftrightarrow$ emission time $t_{\mathrm{em}} \leftrightarrow$ comov. dist. at emission r_{em}
any/all of these denote a cosmic **epoch**

now consider monitoring a dynamical process in a distant source
Q: what would you notice?

Cosmic Time Dilation

when monitoring a distant dynamical process ("standard clock") in addition to redshift will note *duration* change

$$\frac{\delta t_{\rm obs}}{\delta t_{\rm em}} = \frac{a(t_{\rm obs})}{a(t_{\rm em})} = \frac{1+z_{\rm em}}{1+z_{\rm obs}}$$

since cosmic expansion gives $a(t_{\rm obs}) > a(t_{\rm em})$

$\rightarrow$ $\delta t_{\rm obs} > \delta t_{\rm em}$

$\rightarrow$ distant happenings appear in slow motion!

$\rightarrow$ time dilation!

Note: effect depends only on redshift, not on geometry

cosmic time dilation recently observed!

Q: how would effect show up?

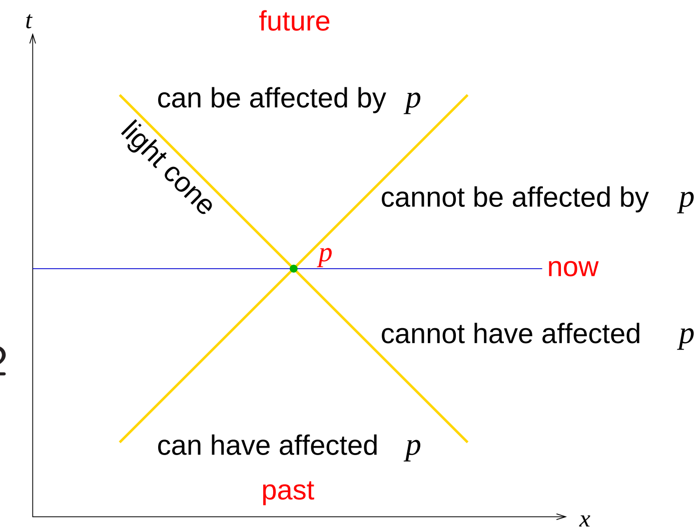
Q: why non-trivial to observationally confirm?

`www: cosmic time dilation evidence`

Cosmic Causality

Recall special relativity (Minkowski space)
$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$

light: $ds = 0$ $\rightarrow$ *cone* $dt^2 = dx^2 + dy^2 + dz^2$



Now RW metric: $ds^2 = dt^2 - a^2 d\ell^2_{\rm com}$
introduce new time variable η: **conformal time**
defined by $d\eta = dt/a(t)$ (see PS2)

$$ds^2 = a(\eta)^2 \left(d\eta^2 - d\ell^2_{\rm com}\right)$$

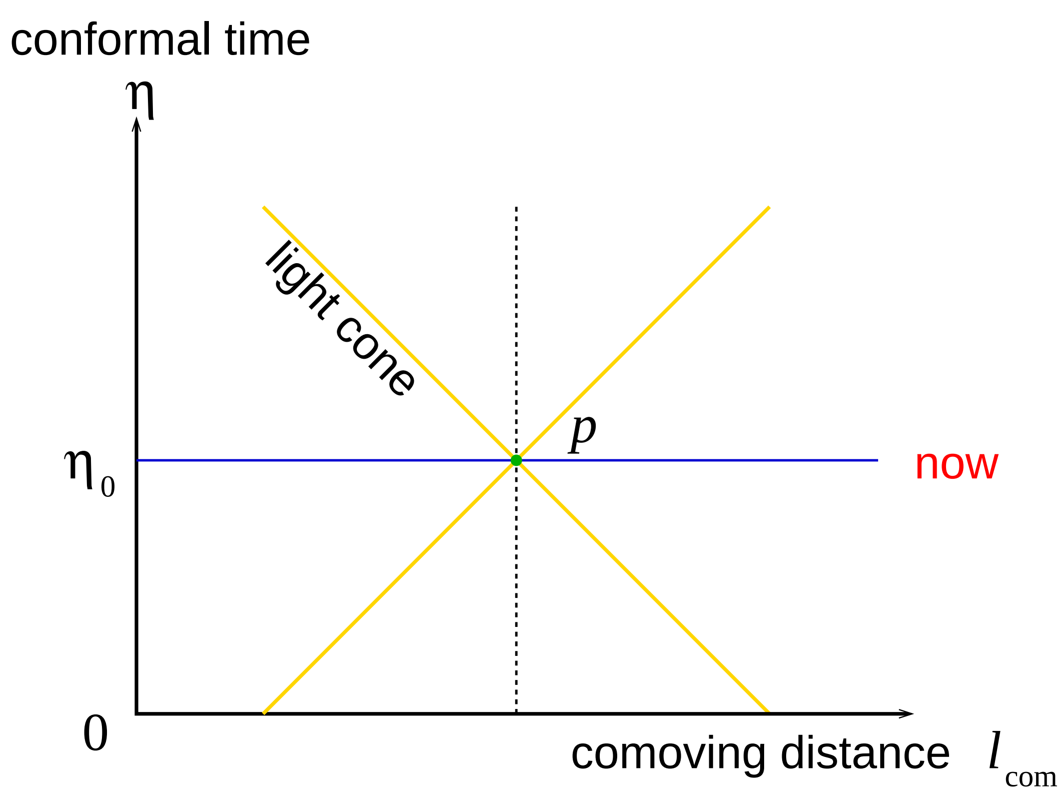
Q: implications?

$$ds^2 = a(\eta)^2 \left(d\eta^2 - d\ell^2_{\text{com}}\right) = a(\eta)^2 \ \times \ \text{(Minkowski structure)}$$

has same features as Minkowski space

$\Rightarrow$ *light cones still defined*

when use comoving lengths and conformal time



For a flat universe ($\kappa = 0$), it's even better:

$$ds^2 = a(\eta)^2 \left(d\eta^2 - dr_{\text{com}}^2\right) = a(\eta)^2 \ \times \ \text{(exact Minkowski form)}$$

In either case $\rightarrow$ spacelike, timelike, lightlike divisions same and in $(\eta, \ell_{\text{com}})$ space:
light cone structure the same $\Rightarrow$ *causal structure the same*!

Namely:

- a spacetime point can only be influenced by events in past light cone
- a spacetime point can only influence events in future light cone

So far: like Minkowski
New cosmic twist: finite cosmic age
Q: implications for causality?

Causality: Particle Horizon

past light cone at t defined by
photon propagation over cosmic history:

$$\int_{t_{\rm em}=0}^{t_{\rm obs}=t_0} \frac{d\tau}{a(\tau)} = \int_0^{r_{\rm em}} \frac{dr}{\sqrt{1-\kappa r^2/R^2}} \equiv d_{\rm hor,com}(t_0)$$

where $d_{\rm hor,com}$ is comoving distance
photon has traveled since big bang

if $d_{\rm hor,com} = \int_0^t d\tau/a(\tau)$ converges
then only a finite part of U has affected us
$\rightarrow$ $d_{\rm hor}$ defines *causal boundary*
$\rightarrow$ "**particle horizon**"

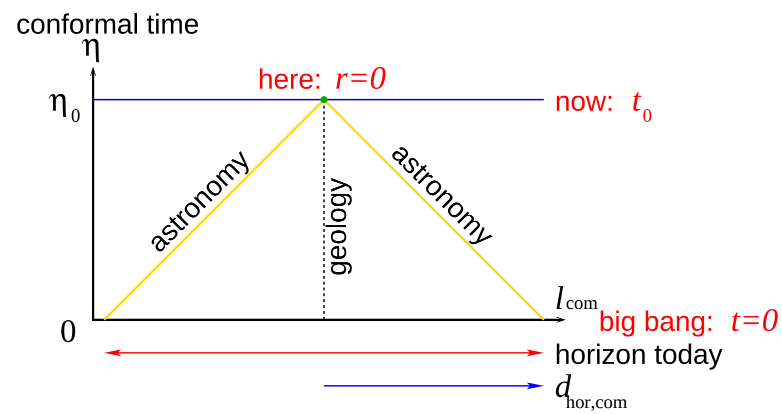
Q: physical implications of a particle horizon?
Q: role of finite age?
Q: sanity check–simple limiting case with obvious result?

Particle Horizons: Implications

our view of the Universe:

⋆ astronomical info comes from events along *past light cone*

⋆ *geological info* comes from *past world line*



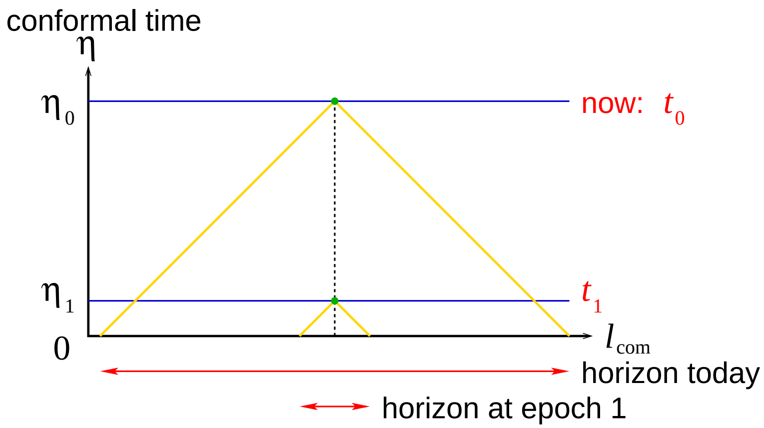
if particle horizon finite (i.e., $\neq \infty$), then $d_{\mathrm{horiz,com}}$:

- gives comoving size of **observable universe**
- encloses region which can communicate over cosmic time $\rightarrow$ causally connected region
- sets "zone of influence" over which particles can "notice" and/or affect each each other and local physical processes can "organize" themselves

e.g., shouldn't see bound structures large than particle horizon!

So *is* d_{hor} finite?
depends on details of $a(t)$ evolution as $t \to 0$:
behavior near singularity crucial

will see in PS3:
for matter, radiation domination:
- d_{hor} finite
- and $d_{\text{hor}} \to 0$ for $t \to 0$



Q: implications for CMB?
Hint: observed $T_{\text{CMB}}(\theta, \phi)$ isotropic to 5th decimal place...

will see in coming weeks
▷ inflation (if real!) adds twist!

Director's Cut Extras

Sketch of Friedmann Derivation in General Relativity

Assume universe mass-energy described by perfect fluid:

- "perfect" $\rightarrow$ no dissipation (i.e., viscosity)
- stress-energy: given density, pressure fields ρ, p
  and 4-velocity field $u_\mu \rightarrow (1,0,0,0)$ for FO

$$
\begin{aligned}
T_{\mu\nu} &= \rho u_\mu u_\nu + p(g_{\mu\nu} - u_\mu u_\nu) \quad (7)\\
&= \mathrm{diag}(\rho, p, p, p)_{\mathrm{FO}} \quad (8)
\end{aligned}
$$

Recall: stress-energy conservation is

$$\nabla_\nu T^{\mu\nu} = 0 \quad (9)$$

where ∇_μ is covariant derivative
For RW metric, this becomes:

$$d(a^3\rho) = -p \; d(a^3) \quad (10)$$

1st Law of Thermodynamics!

Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{11}$$

Given RW metric (orthogonal, max symmetric):

- *Q: how many nonzero Einstein eqs generally? here?*
- *Q: what goes into $G_{\mu\nu}$? what will this be for RW metric?*

Einstein eq:

$G_{\mu\nu}, T_{\mu\nu}$ symmetric 4×4 matrices $\rightarrow$ 10 independent components

in general, Einstein $\rightarrow$ 10 equations

but cosmo principle demands: space-time terms $G_{0i} = 0$

and off-diagonal space-space $G_{ij} = 0$

else pick out special direction $\Rightarrow$ only diagonal terms nonzero

and all 3 "p" equations same

Einstein $\rightarrow$ two independent equations

$$
\begin{aligned}
G_{00} &= 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} &(12)\\
&= 8\pi G T_{00} = 8\pi G\rho &(13)\\
G_{ii} &= 6\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3\kappa}{R^2a^2} &(14)\\
&= 8\pi G T_{ii} = 8\pi G p &(15)
\end{aligned}
$$

After rearrangement, these become
the Friedmann "energy" and acceleration equations!