Astro 507

Lecture 22

March 23, 2020

Announcements:

- Welcome to Shelter in Place Cosmology!
- Lecture in real time on Zoom, video posted after
- **Problem Set 4 due Friday**

Today:

- Cosmology in quarantine–how we will muddle through
- the CMB and cosmic recombination

Checking In

We are in stressful uncharted territory

Most importantly, I want you to be safe and healthy

If you have problems, let me know!

We will figure out how to get through this

Course Revisions

Modified Course Requirements

We will proceed with the same material
But now we will revise the requirements:

- *drop the lowest PS score*
  replace it with the average of the other PS scoes

- *drop the lowest Preflight score*
  replace it with the average of the other preflights

Modified Course Meetings

on Zoom, at the usual time
I can't see your faces! Turns out that's useful feedback
Please ask questions: unmute and speak, or use chat

Before Break

Last time: theory of isotropic CMB spectrum
key aspect: the *only* process acting is Thompson scattering
photon scattering on free electrons $e^- \gamma \to e^- \gamma$

Given a photon spectrum I_ν *prior* to decoupling
Q: what is spectrum after Thompson freezeout?

Observed (post-decoupling) CMB spectrum: *thermal*
Q: implications?

Q: what physically controls onset of decoupling?

The CMB Demands a Hot Big Bang

observe *thermal* (Planck) CMB spectrum today
$\Rightarrow$ *thermal* CMB spectrum *pre*-decoupling!
$\Rightarrow$ in early U: photons thermalized, coupled to matter!

Cosmic matter & radiation once in "good thermal contact"
$\rightarrow$ but this requires much higher T, ρ than seen today
$\rightarrow$ CMB demands Universe went through *hot*, *dense* early phase
$\Rightarrow$ CMB $\rightarrow$ *hot big bang*

Compton/Thomson scattering conserves photon number
but Planck spectrum has fixed number density at T
$\Rightarrow$ **early Universe needed photon number-changing processes**
e.g., bremsstrahlung $e +$ nucleus $\rightarrow e +$ nucleus $+ \gamma$

The Ratio of Baryons to Photons

The number of barons per photon
is the "baryon-to-photon ratio" $\eta \equiv n_B / n_\gamma$

photons not conserved in general:
e.g., Bremsstrahlung $e \to e + \gamma$
so chem pot $\mu_e = \mu_e + \mu_\gamma \to \mu_\gamma = 0$
$\to n_\gamma \sim T^3$: fixed by T alone

baryons conserved:
#baryons = const in comoving vol
$d(n_B a^3) = 0 \to n_B \propto a^{-3}$
$\to$ so $\mu_B(T) \neq 0$ enforces this scaling

Thus we have

$$\eta = \frac{n_{B,0} a^{-3}}{n_{\gamma,0} (T/T_0)^3} = \left(\frac{T_0}{a T} \right)^3 \eta_0 \tag{1}$$

baryon number conservation: $n_{\mathrm{B}} \propto a^{-3}$
thermal photons: $n_\gamma \propto T^3$

so as long as $T \sim 1/a$ then
$\eta = const$! *baryon-to-photon ratio conserved!*
thus we expect $\eta_{\mathrm{BBN}} = \eta_{\mathrm{CMB}} = \eta_0$!

numerically (from BBN, CMB anisot):

$$\frac{n_{\mathrm{b}}}{n_\gamma} = \eta_0 \sim 6 \times 10^{-10} \ll 1 \quad (2)$$

$$n_\gamma \sim 10^9 \, n_{\mathrm{b}} \quad (3)$$

huge number of photons per baryon! never forget!
$\Rightarrow$ early U had huge photon source! *Q: ideas?*

but note $\rho_B/\rho_\gamma \sim m_B n_B / T n_\gamma \sim \eta m_B / T \neq const$

Last Scattering: Including Recombination

Recombination $p + e^- \to H + \gamma$

For simplicity, we will assume baryons are only protons
and will consider only Thomson scattering (excellent approx!)

`www: laboratory hydrogen plasma`

Then: scattering rate per photon is

$$\Gamma_\gamma = n_{e,\text{free}} \, \sigma_T \, c \quad \propto \quad n_{e,\text{free}} \tag{4}$$

and last scattering when $\Gamma_\gamma \simeq H$

last scattering/decoupling controlled by **free electron density**

$n_{e,\text{free}}$ changes due to

∞ • cosmic volume expansion $\propto a^{-3}$
• recombination: free e^- lost to neutral H

rewrite to account for each $n_{e,\text{free}}$ effect separately:

$$n_{e,\text{free}} = X_e n_{e,\text{tot}} = X_e n_{\text{baryon}} \tag{5}$$

- baryon density $n_{\text{b}} \propto a^{-3} \propto T^3$

gives volume dilution

- "ionization fraction"

$$X_e \equiv \frac{n_{e,\text{free}}}{n_{e,\text{free}} + n_{e,\text{bound}}} = \frac{n_p}{n_p + n_{\text{H}}} = \frac{n_p}{n_{\text{b}}} \tag{6}$$

unchanged by volume dilution

only depends on recombination thermodynamics:

i.e., $X_e = X_e(T) = X_e(z)$

Q: limiting values of X_e at high and low T?

Q: naïve estimate of recombination $T_{\text{rec}}, z_{\text{rec}}$?

Q: zeroth-order treatment of $X_e(T)$?

Recombination: Naïve View

Given hydrogen binding energy

$$B_{\rm H} = E(p) + E(e) - E({\rm H}) = 13.6 \text{ eV}$$

simple estimate of recomb epoch goes like this:

Binding sets energy scale, so
⋆ when particle energies above $B_{\rm H}$: U ionized,
⋆ otherwise: U neutral
$\rightarrow$ naïvely expect transition at $T_{\rm rec,naive} = B_h \sim 150,000$ K

But we know $T = T_0/a$, so estimate recomb at

$$\left.\begin{aligned} a_{\rm rec,naive} &= \frac{T_0}{T_{\rm rec,naive}} &&\sim 2 \times 10^{-5} \\ z_{\rm rec,naive} &= \frac{T_{\rm rec,naive}}{T_0} - 1 &&\sim 50,000 \end{aligned}\right\} \text{wrong!}$$

Q: guesses as to what's wrong?
Q: how to do this right?

Recombination: Equilibrium Thermodynamics

dominant cosmic plasma components γ, p, e, H (ignore He, Li)
equilibrium: equal forward and reverse rates for

$$p + e \leftrightarrow \mathrm{H} + \gamma$$

and so chem potentials have

$$\mu_p + \mu_e = \mu_{\mathrm{H}} \tag{7}$$

Extras today: for non-rel species $n = g(mT/2\pi\hbar^2)^{3/2} e^{-(m-\mu)/T}$
thus we have **Saha equation**

$$\frac{n_e n_p}{n_{\mathrm{H}}} = \frac{g_e g_p}{g_{\mathrm{H}}} \left(\frac{m_e m_p}{m_{\mathrm{H}}}\right)^{3/2} \left(\frac{T}{2\pi\hbar^2}\right)^{3/2} e^{-(m_e+m_p-m_{\mathrm{H}})/T} \tag{8}$$

$$\approx \left(\frac{m_e T}{2\pi\hbar^2}\right)^{3/2} e^{-B/T} \tag{9}$$

where $B \equiv m_e + m_p - m_{\mathrm{H}} = 13.6$ eV: binding energy introduce "free electron fraction" $X_e = n_e/n_B$

use $n_B = \eta n_\gamma \propto \eta T^3$

from Extras: $n_\gamma = 2\zeta(3)/\pi^2 \, T^3$, with $\zeta(3) = \sum_1^\infty 1/n^3 = 1.20206\ldots$

and note that $n_p = n_e$ *Q: why?*, so

$$\frac{n_e^2}{n_{\rm H} n_B} = \frac{X_e^2}{1 - X_e} = \frac{\sqrt{\pi}}{4\sqrt{2}\zeta(3)} \frac{1}{\eta} \left(\frac{m_e}{T}\right)^{3/2} e^{-B/T} \tag{10}$$

Q: sanity checks? what sets characteristic T scale?

Q: when is $X_e = 0$ (exactly)?

At last–recombination!

Q: how define physically?

Q: how define operationally, in terms of X_e?

Q: given some $X_{e,\rm rec}$, how to get $z_{\rm rec}$?

The Epoch of Recombination

Saha gives

$$\frac{1-X_e}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\pi^{1/2}} \, \eta \, \left(\frac{B}{m_e}\right)^{3/2} \left(\frac{T}{B}\right)^{3/2} e^{B/T} \tag{11}$$

if always equilib, then strictly $X_e = 0$ only at $T = 0$
but note $e^{B/T}$: X_e exponentially small when $T \ll B$

viewed as a function of $B/T \equiv u$

$$\frac{1-X_e}{X_e^2} = \frac{4\sqrt{2}\zeta(3)}{\pi^{1/2}} \, \eta \, \left(\frac{B}{m_e}\right)^{3/2} u^{3/2} e^u \equiv A \, u^{3/2} e^u \tag{12}$$

where $A = 4\sqrt{2}/\pi^{1/2}\zeta(3) \, \eta \, (B/m_e)^{3/2}$

Q: what is order-of-magnitude of A?
Q: implications for recombination?
Q: physical picture?

in recombination Saha expression $(1 - X_e)/X_e = A(B/T)^{3/2} e^{B/T}$
prefactor is tiny!
$A \sim \eta(B/m_e)^{3/2} \sim 10^{-9}(10^{-5})^{3/2} \sim 10^{-16}$!
why? largely due to *tiny baryon-to-photon ratio*

but when recombine: $1 - X_e \simeq X_e$
so require $1 \sim 10^{-16}(B/T_{\rm rec})^{3/2} e^{B/T_{\rm rec}}$
$\Rightarrow$ so need $B/T_{\rm rec} \gg 1$ to offset prefactor
$\Rightarrow$ and thus $T_{\rm rec} \ll B$!

more carefully define recomb: $X_e = X_{e,{\rm rec}} = 0.1$
(arbitrary, but not crazy; see PS4)
then solve for $T_{\rm rec}$:

$$\frac{B}{T_{\rm rec}} = \ln\left(\frac{\pi^{1/2}}{4\sqrt{2}\zeta(3)}\right) + \ln\left(\frac{1 - X_{e,{\rm rec}}}{X_{e,{\rm rec}}^2}\right) + \ln \eta^{-1} + \frac{3}{2}\ln\frac{m_e}{B} - \frac{3}{2}\ln\frac{B}{T}$$
$$\sim \quad 40 \quad (\gg 1)$$

(ignore or iterate $\ln B/T$ term)

Recombination Quantified

and so

$$T_{\text{rec}} \approx \frac{B}{40} \simeq 0.3 \text{ eV} \ll B \quad (13)$$

$$z_{\text{rec}} \approx 1400 \ll z_{\text{rec,naive}} \quad (14)$$

$$t_{\text{rec}} \approx \frac{2}{3\sqrt{\Omega_{\text{m}}}} H_0^{-1}(1+z_{\text{rec}})^{-3/2} = 350,000 \text{ yrs} \quad (15)$$

PS4: try it yourself!

Implications for CMB frequency spectrum:

- at recomb: emission lines created at $h\nu_{\text{rec}} \gtrsim 3B/4$ and thus at $h\nu_{\text{rec}} \gtrsim 30kT_{\text{rec}}$
- post-recomb: T and ν both redshift the same way, so
- CMB spectrum *distorted* from Planck at high freq: $h\nu \gtrsim 30kT$
- small signal, difficult to observe, but tantalizing `www: predictions`

Q: what physically is responsible for $T_{\text{rec}} \ll B$?

Recombination "Delay"

Why is $T_{\text{rec}} \ll B$?

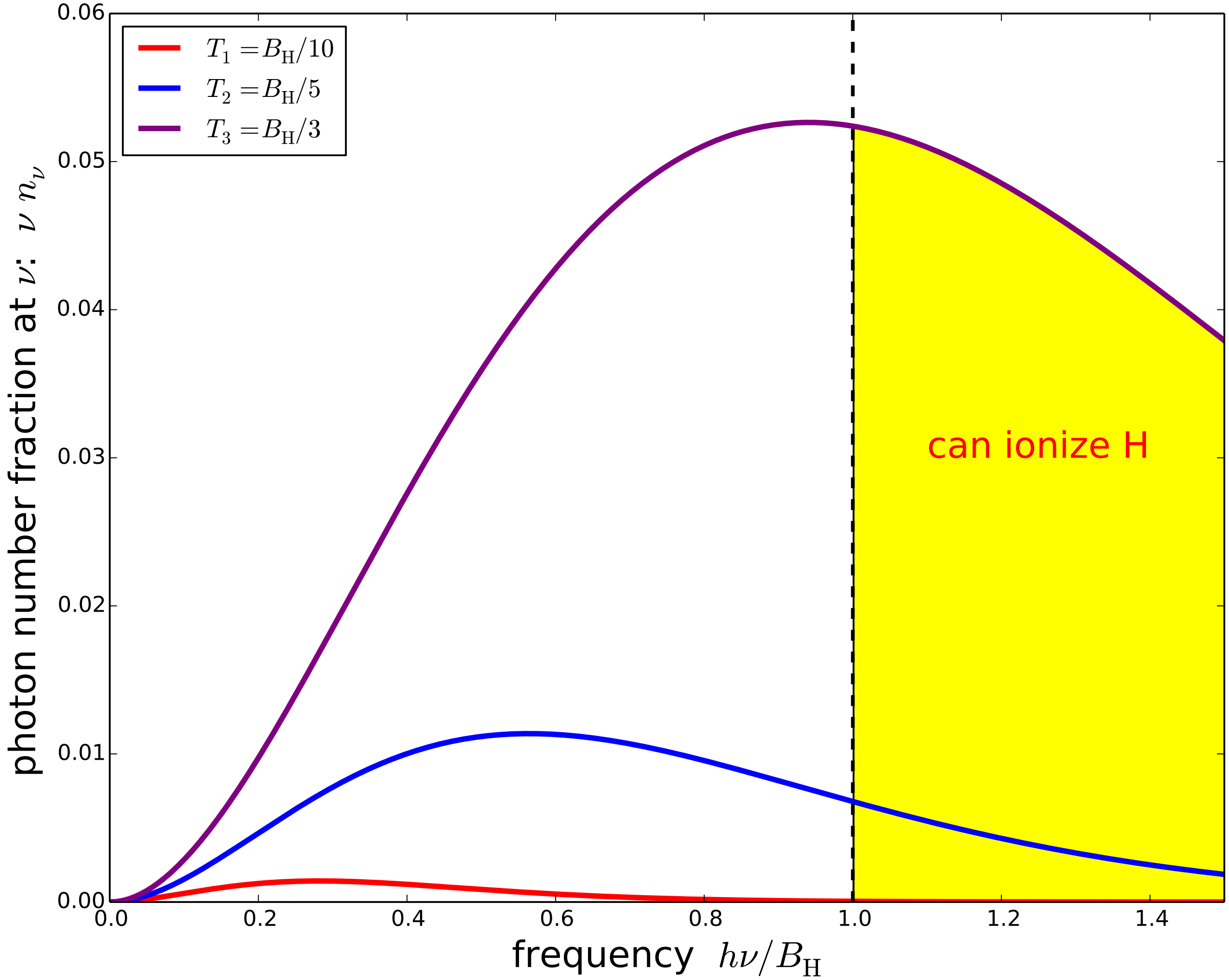
▷ because for small X_e, Saha says $X_e \propto 1/\eta^{1/2} \gg 1$

▷ many photons per baryon: even if typically $E_\gamma \ll B$,
high-E tail of Planck distribution not negligible (at first)
lots of ionizing photons with $E_\gamma \geq B$
H dissociated as soon as formed

When does dissociation stop?
can show that fraction of photons with $E_\gamma > B$
is roughly $f_{\text{ionizing}} \sim e^{-B/T}$
so ratio of ionizing photons per baryon is

$$\frac{n_{\gamma,\text{ionizing}}}{n_B} \sim \frac{e^{-B/T}}{\eta} \quad (16)$$

estimate recombination when $n_{\gamma,\text{ionizing}}/n_B \sim 1$
$\rightarrow T \sim B/\ln \eta^{-1} \ll B$ (check!)
$\Rightarrow$ recombination "delayed" to huge photon-to-baryon ratio



$T_1 = B_\mathrm{H}/10$
$T_2 = B_\mathrm{H}/5$
$T_3 = B_\mathrm{H}/3$
can ionize H
photon number fraction at ν: $\nu\, n_\nu$
frequency $h\nu/B_\mathrm{H}$
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

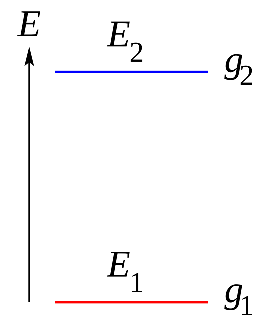
Director's Cut Extras

Statistical Mechanics and Cosmology

For much of cosmic time contents of U. in *thermal equilibrium*

statistical mechanics: at fixed $T \rightarrow$ matter & radiation n, ρ, P
then cosmic $T(a)$ evolution $\rightarrow n, \rho, P$ at any epoch

Boltzmann: consider a particle (elementary or composite)
with a series of energy states:
for two sets of states with energies E_1 and $E_2 > E_1$
and degeneracies (# states at each E) g_1 and g_2
ratio of number of particles in these states is



$$\frac{n(E_2)}{n(E_1)} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/T}$$

where I put $k = 1$, i.e., $kT \rightarrow T$

Example: atomic hydrogen, at T
Q: ratio of ground ($1S$) *to 1st excited state* ($2P$) *populations?*

Atomic hydrogen (H I):

- energy levels: $E_n = -B_{\mathrm{H}}/n^2$ for $n \geq 1$
- angular momenta degeneracies: $g_\ell = 2\ell + 1$

1S: $n = 1 \rightarrow E(1S) = -B$; $\ell = 0 \rightarrow g(1S) = 1$

2P: $n = 2 \rightarrow E(2P) = -B/4$; $\ell = 1 \rightarrow g(2P) = 3$

$$\frac{n(2P)}{n(1S)} = 3e^{-3B/4T} = 3e^{-120{,}000\,\mathrm{K}/T} \tag{17}$$

Q: sanity checks–is this physically reasonable?

Q: how does this ratio change if plasma is partially ionized i.e., contains both H I *and* H II$=$ H$^+ = p$*?*

Note: H is bound system $\rightarrow$ discrete energies
we now broaden analysis to include unbound systems
$\rightarrow$ continuous energies, momenta

Statistical Mechanics in a Nutshell

classically, **phase space** $(\vec{x}, \vec{p})$
completely describes particle state
but quantum mechanics $\rightarrow$ uncertainty $\Delta x \Delta p \geq \hbar/2$
semi-classically: min phase space "volume"
$(dx\ dp_x)(dy\ dp_y)(dz\ dp_z) = h^3 = (2\pi\hbar)^3$
per quantum state of fixed $\vec{p}$

define "**occupation number**" or "**distribution function**" $f(\vec{x}, \vec{p})$:
number of particles in each phase space "cell"
Q: f range for fermions? bosons?

$$dN = g f(\vec{x}, \vec{p}) \frac{d^3\vec{x}\ d^3\vec{p}}{(2\pi\hbar)^3} \tag{18}$$

where g is # internal (spin/helicity) states:
Q: $g(e^-)$? $g(\gamma)$? $g(p)$?

Fermions: $0 \le f \le 1$ (Pauli)
Bosons: $f \ge 0$ $g(e^-) = 2s(e^-) + 1 = 2$ electron, same for p
$g(\gamma) = 2$ (polarizations) photon

Particle phase space occupation f determines bulk properties

Number density

$$n(\vec{x}) = \frac{d^3N}{d^3x} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ f(\vec{p}, \vec{x}) \tag{19}$$

Mass-energy density

$$\varepsilon(\vec{x}) = \rho(\vec{x})c^2 = \langle En \rangle = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ E(p)\ f(\vec{p}, \vec{x}) \tag{20}$$

Pressure see director's cut extras for more

$$P(\vec{x}) = \langle p_i v_i n \rangle_{\text{direction}\,i} \overset{\text{isotrop}}{=} \frac{\langle pvn \rangle}{3} = \frac{g}{(2\pi\hbar)^3} \int d^3\vec{p}\ \frac{p\,v(p)}{3}\ f(\vec{p}, \vec{x}) \tag{21}$$

Q: these expressions are general–simplifications in FLRW?

FRLW universe:

- homogeneous $\rightarrow$ no $\vec{x}$ dep
- isotropic $\rightarrow$ only $\vec{p}$ magnitude important $\rightarrow$ $f(\vec{p}) = f(p)$

in **thermal equilibrium** at T:

▷ *Boson occupation number*

$$f_{\mathrm{b}}(p) = \frac{1}{e^{(E-\mu)/kT} - 1} \tag{22}$$

▷ *Fermion occupation number*

$$f_{\mathrm{f}}(p) = \frac{1}{e^{(E-\mu)/kT} + 1} \tag{23}$$

Note: μ is "*chemical potential*" or "Fermi energy"
$\mu = \mu(T)$ but is *independent* of E

If $E - \mu \gg T$: both $f_{\mathrm{f,b}} \longrightarrow f_{\mathrm{Boltz}} = e^{-(E-\mu)/kT}$
$\rightarrow$ *Boltzmann distribution*

The Meaning of the Chemical Potential

For a particle species in thermal equilibrium

$$f(p; T, \mu) = \frac{1}{e^{[E(p)-\mu]/kT} \pm 1} \tag{24}$$

What is μ, and what does it mean physically?

First, consider what if $\mu = 0$?

- then f depends only on T and particle mass and thus so do n, ρ, P *Q: why?*
- all samples of a substance at fixed T have exactly the same n, ρ, P!
- and hotter $\rightarrow$ larger n, ρ, P

sometimes true! *Q: examples?*
but not always! *Q: examples?*

Q: what is physics behind μ?

Chemical Potential & Number Conservation

particle number often *conserved*
$\rightarrow n = n_{\text{cons}}$ fixed by initial conditions, not T

if particle number conserved, then $\mu \neq 0$ and μ determined by solving $n_{\text{cons}} = n(\mu, T) \rightarrow \mu(n_{\text{cons}}, T)$

so: $\mu \neq 0 \ \Leftrightarrow$ particle number conservation

Chemical Potential and Reactions

reactions change particle numbers among species

in "chemical" equilibrium: forward rate = reverse rate
for example: "two-to-two" reaction $a + b \leftrightarrow A + B$

conservation laws (charge, baryon number, etc.)
force relations between chemical potentials
so in above example: $\mu_a + \mu_b = \mu_A + \mu_B$
sum of chemical potentials "conserved"

in general:

$$\sum_{\text{initial particles}\, i} \mu_i = \sum_{\text{final particles}\, f} \mu_f \tag{25}$$

Equilibrium Thermodynamics

Gas of mass m particles at temp T:
n, ρ, and P in general complicated
because of $E(p) = \sqrt{p^2 + m^2}$
but simplify in ultra-rel and non-rel limits

Non-Relativistic Species

$E(p) \simeq m + p^2/2m$, $T \ll m$
for $\mu \ll T$: Maxwell-Boltzmann, same for Boson, Fermions

for non-relativistic particles = *matter*
energy density, number density vs T?
Q: recall $n(a), \rho(a)$ *and* $T(a)$*?*

Non-Relativistic Species

number density

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} \int d^3p \; e^{-p^2/2mkT} \tag{26}$$

$$= g e^{-(mc^2-\mu)/kT} \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \tag{27}$$

energy density:

$$\rho c^2 = \langle En \rangle = \varepsilon_{\text{rest mass}} + \varepsilon_{\text{kinetic}} \tag{28}$$

$$= mc^2\, n + \frac{3}{2}\, kT\, n \tag{29}$$

$$\simeq \varepsilon_{\text{rest mass}} = mc^2 n \tag{30}$$

pressure

$$P = \frac{\langle pvn \rangle}{3} = \frac{\left\langle p^2 n/m \right\rangle}{3} = \frac{2}{3}\varepsilon_{\text{kinetic}} \tag{31}$$

$$= nkT \ll \rho c^2 \tag{32}$$

recover the ideal gas law!

Kinetic Theory of Pressure due to Particle Motions

consider cubic box, sidelength L (doesn't really need to be cubic)
contain "gas" of N particles: can be massive or massless
particles collide with walls, bounce back elastically
particles exert force on wall $\leftrightarrow$ wall on particles
this lead to bulk *pressure*

focus on one particle, and its component of motion
in one (arbitrary) axis x: speed v_x, momentum p_x

- *elastic* collision: $p_{x,init} = -p_{x,fin} \rightarrow \delta p_x = 2p_x$
- collision time interval for same wall: $\delta t_x = v_x/2L$
- single-particle *momentum transfer* (force) per wall:
  $F_x = \delta p_x/\delta t_x = p_x v_x/L$
- *single-particle force* per wall area:
  $P = F_x/L^2 = p_x v_x/L^3 = p_x v_x/V$

Q: total pressure?

total pressure is sum over all particles:

$$P = \sum_{\text{particles}} \sum_{\ell=1}^{N} \frac{p_x^{(\ell)} v_x^{(\ell)}}{V} \tag{33}$$

can rewrite in terms of an average momentum flux

$$P = \frac{N}{V} \frac{\sum_{\ell=1}^{N} p_x^{(\ell)} v_x^{(\ell)}}{N} = \langle p_x v_x \rangle \, n \tag{34}$$

where $n = N/V$ is *number* density

$\langle p_x \rangle \, n$ would be average *momentum density* along x

and $\langle p_x v_x \rangle \, n$ is average *momentum flux* along x

if particle gas has isotropic momenta, then

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_x \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \langle p v \rangle \tag{35}$$

so $\boxed{P = \frac{1}{3} \langle p v \rangle \, n}$

Ultra-Relativistic Species

$E(p) \simeq cp \gg mc^2$ (i.e., $kT \gg mc^2$):
Also take $\mu = 0$ ($\mu \ll kT$)

energy density, number density?
Q: recall the answers?

for relativistic bosons

number density

$$n_{\mathrm{rel,b}} = \frac{g}{(2\pi\hbar)^3} \int d^3p \; \frac{1}{e^{cp/kT} - 1}$$

$$= \frac{4\pi g}{(2\pi\hbar)^3} \int dp \;\, p^2 \; \frac{1}{e^{cp/kT} - 1} = \frac{g}{2\pi^2} \; \left(\frac{kT}{\hbar c}\right)^3 \; \int_0^\infty du \;\, u^2 \; \frac{1}{e^u - 1}$$

$$= g \frac{\zeta(3)}{\pi^2} \; \left(\frac{kT}{\hbar c}\right)^3 \propto T^3$$

where

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.20206\ldots \qquad (36)$$

relativistic fermions:

$$n_{\rm rel,f} = \frac{3}{4} n_{\rm rel,b} \tag{37}$$

so $n \propto T^3$ for both

e.g., CMB today: $n_{\gamma,0} = 411\ {\rm cm}^{-3}$

energy density: relativistic bosons

$$\begin{aligned} \rho_{\rm rel,b} c^2 &= \frac{g}{(2\pi\hbar)^3} \int d^3p\ cp\ \frac{1}{e^{cp/kT} - 1} \\ &= \frac{g}{2\pi^2} \frac{(kT)^4}{(\hbar c)^3} \int_0^\infty du\ u^3\ \frac{1}{e^u - 1} \\ &= g \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3} \end{aligned}$$

and for fermions

$$\rho_{\rm rel,f} = \frac{7}{8} \rho_{\rm rel,b} \tag{38}$$

so $\rho \propto T^4$ for both

pressure

$$P_{\text{rel}} = \left\langle \frac{pv}{3} n \right\rangle = \frac{1}{3} \rho_{\text{rel}} c^2 \tag{39}$$

since $v = c$

$P \propto T^4$

Temperature Evolution

If in therm eq, maintain photon occ. #

$$f(p) = \frac{1}{e^{p/T} - 1} \tag{40}$$

but $cp = h\nu = hc/\lambda \propto 1/a(t)$:
$\Rightarrow p = p_0/a$

w/o interactions, const # γ per mode p
$\Rightarrow f(p) = const$
$\Rightarrow p(t)/T(t) = p_0/T_0$
$\Rightarrow \boxed{T/T_0 = p/p_0 = 1/a = 1 + z}$
e.g., at $z = 3$, CMB $T = 4T_0 \simeq 11$ K
(measured in QSO absorption line system!)

recall: used $w = 1/3$ to show $\rho_\gamma \propto a^{-4}$
but blackbody $\rho_\gamma \propto T^4$
together $T \propto 1/a$ (OK!)