Astro 507

Lecture 31

April 13, 2020

Announcements:

- **Problem Set 5 due today**
- **Preflight 6: Part (a) Due Friday April 17**
  Wikipedia Cosmology! propose a wikipedia upgrade
  can be modest and targeted, or more ambitious

Last time: slow-roll inflation
scalar field dynamics in an expanding universe
Q: what is ϕ*?* $V(\phi)$*?*
Q: what is needed for ϕ *to inflate the universe?*

inflation: let there be scalar field ϕ

minimal version–self-coupled: $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$

cosmic equation of motion $\ddot{\phi} + 3H\dot{\phi} + dV/d\phi = 0$

initial conditions:

(a) ϕ dominates cosmic energy density $\rho_{\rm tot} \approx \rho_\phi$

(b) ϕ away from ground state

(c) potential term dominates over kinetic: $\rho_\phi \approx V(\phi)$

result:

(a) ϕ controls cosmic dynamics: $H^2 = (\dot{\phi}^2/2 + V)/3m_{\rm pl}^2$

(b) $V(\phi) > 0$: vacuum energy fills the universe

(c) $w_\phi \rightarrow -1$: exponential expansion!

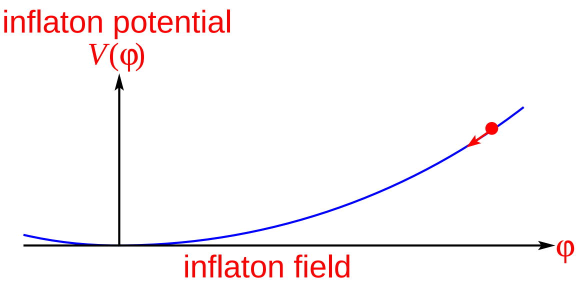
Q: what is needed to ensure (c)?

To inflate, need slow ϕ evolution:

$\ddot{\phi} \ll 3H\dot{\phi}$ $\leftrightarrow$ friction large:

$\Rightarrow$ achieve "terminal speed"

$$\dot{\phi} \approx -\frac{1}{3H} V'$$



slowness imposes conditions on the potential $V(\phi)$:

$$\epsilon(\phi) = \frac{m_{\rm pl}^2}{2} \left(\frac{V'}{V} \right)^2 \ll 1 \quad (1)$$

$$\eta(\phi) = m_{\rm pl}^2 \frac{V''}{V} \ll 1 \quad (2)$$

small derivatives $\rightarrow$ **potential must be flat**

note potential "curvature" scale is $m_{\rm pl}$: Planck!

The Energy Scale of Inflation

generically expect $\phi \gtrsim m_{\rm pl} \sim 10^{18}$ GeV!

$\Rightarrow$ for successful inflation, field probes the Planck scale (?)

;-) a good thing?

hints at quantum gravity

if $\Omega_{\rm init} \gtrsim 1$, inflation prevents U. collapse $\rightarrow$ black hole

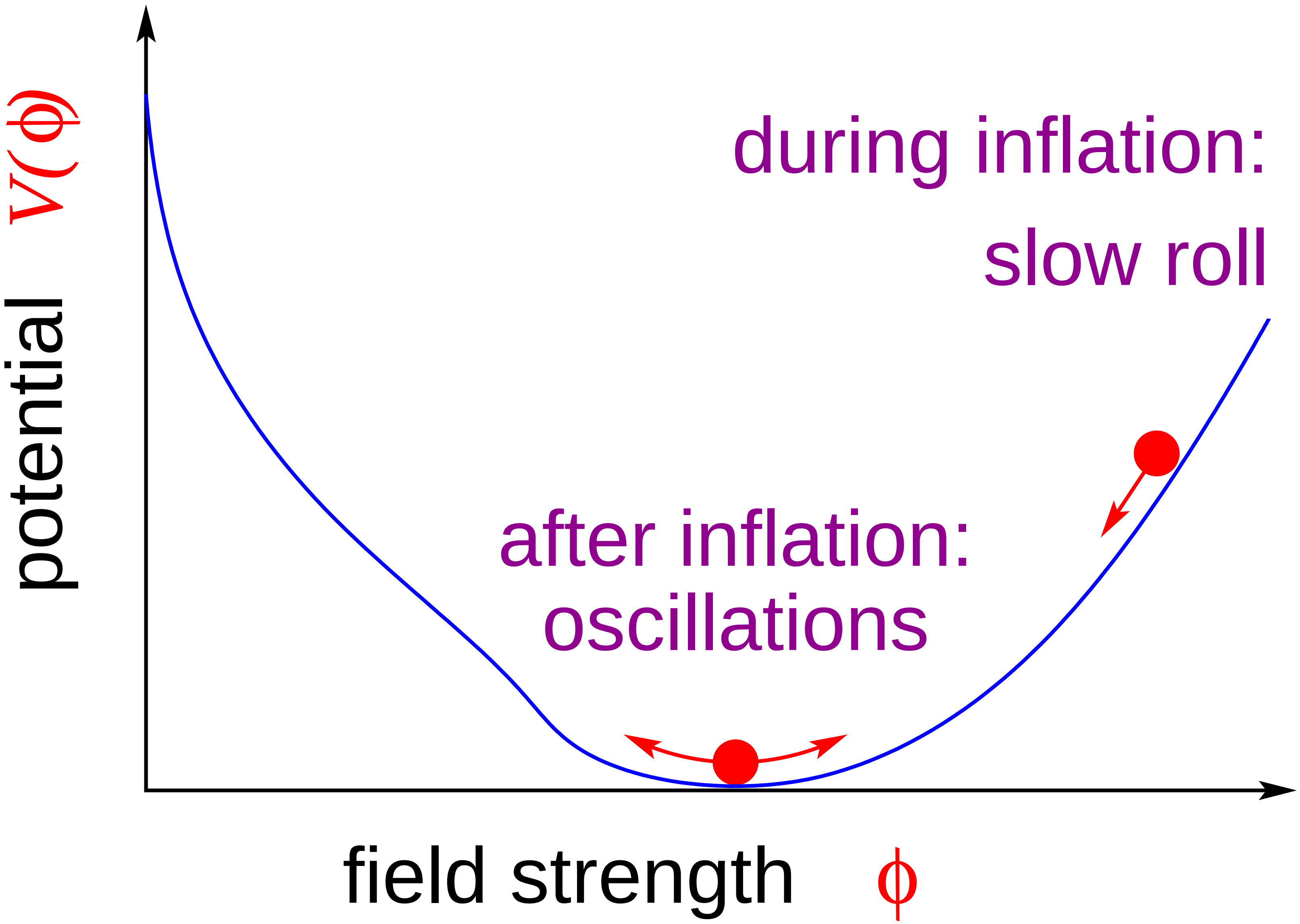
=:-o a bad thing?

quantum gravity a prerequisite for inflation models?

moves away Guth's original idea, GUT physics?

★ ϵ, η also can quantify conditions for *ending* inflation

Q: what conditions needed to end inflation?


potential V(φ)
during inflation:
slow roll
after inflation:
oscillations
field strength φ

A Graceful Exit from Inflation

inflaton continues until acceleration stops ($w_\phi > 0$)
$\rightarrow$ potential energy no longer dominates cosmic ρ
all matter and radiation inflated away, so "rescue" comes from kinetic energy $\dot{\phi}^2/2$ (by itself, has $w = +1$!)

in terms of potential, exit when slow roll stops
quantified by slow-roll parameters
i.e., ϕ evolves until $\epsilon(\phi) \sim 1$

inflaton requirements:
- to achieve slow roll $\rightarrow$ need flat V far from minimum
- to end slow roll $\rightarrow$ need non-flat $V' \gtrsim V/m_{\rm pl}$ approaching minimum

Q: and then...? What's the Universe like? What happens next?

Reheating: Back to the Hot Big Bang

After $e^{60} \sim 10^{26}$ expansion
radiation, matter particles diluted to negligibility as a^{-3}
temperature drop $T \sim 1/a \to 0$: "supercooling"

But since $V(\phi) \sim const$ during inflation
inflaton energy density still large afterwards
must convert to hot, radiation-dominated early U: **reheating**

Details complicated, model-dependent; basic idea:
⋆ ϕ evolves in non-inflationary way
⋆ quantum effects drive energy conversion

Inflation and the Rest of Cosmology

Reheating Temperature

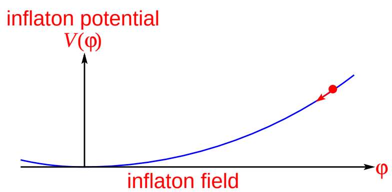
⋆ All of ‘usual” hot big bang begins *after* reheat

⋆ Must reheat enough for U to undergo
any and all known hot big phases
e.g., have to *at least* heat up to have nucleosynthesis
i.e., successful nuke requires $T_{\text{reheat}} > 1$ MeV
earlier phases (if any) demand hotter reheat

Ingredients of an Inflationary Scenario

Recipe:

1. **inflaton field ϕ must exist** in early U.
2. must have $\rho_\phi \approx V$ so that $w_\phi \to -1$
   so that $a \sim e^{Ht}$
3. continue to exponentiate $a \sim e^{N} a_{\text{init}}$
   for at least $N = \int H\, dt \gtrsim 60$ e-folds



4. stop exponentiating eventually (**"graceful exit"**)
5. convert field ρ_ϕ back to radiation, matter (**"reheating"**)
6. then ϕ must "keep a low profile," $\rho_\phi \ll \rho_{\text{tot}}$

7 (bonus) what about acceleration and dark energy today?

Q: what can we say about how inflation fits
in the sequence *of cosmic events, e.g.*
monopole production, baryon genesis, BBN, CMB?

Cosmic Choreography: The Inflationary Tango

Inflation must occur such that it
solves various cosmological problems, then
allows for the universe of today, which *must*

- ▷ contain the known particles, e.g., a net baryon number
- ▷ pass thru a radiation-dominated phase (BBN)
  and a matter-dominated phase (CMB, structure formation)

$\Rightarrow$ this forces an ordering of events

Cosmic Choreography: Required *time-ordering*

1. monopole production (if any)
2. inflation
3. baryogenesis (origin of $\eta \neq 0$)
4. radiation $\rightarrow$ matter $\rightarrow$ dark energy eras

Electroweak woes: hard to arrange baryogenesis afterwards!

Intermission: Questions?

Inflation, Inhomogeneities, and Quantum Mechanics

Thus far: classical treatment of inflaton field

(except for inflaton decays during reheating)

- ϕ described by classical equations of motion
- taken to hold for arbitrarily small ϕ

In this picture:

when inflation ends, universe essentially

▷ perfectly flat, and

▷ perfectly smooth—i.e., density spatially uniform

regardless of initial conditions (as long as they allowed inflation)

Classical Inflation and Smoothness

expect initial spatial inhomogeneities in $\phi(\vec{x})$
but evolves classically as

$$\ddot{\phi} - \nabla^2 \phi + 3H\dot{\phi} - V' = 0 \tag{3}$$

where

$$\nabla^2 = \sum \frac{\partial^2}{\partial x^2_{\rm phys}} = \frac{1}{a^2} \sum \frac{\partial^2}{\partial x^2_{\rm com}} \tag{4}$$

inhomogeneities $\delta\phi(\vec{x})$ measured by nonzero gradients
but since $\nabla^2 \propto 1/a^2 \rightarrow 0$ exponentially, classically: $\delta\phi(\vec{x}) \rightarrow 0$
$\Rightarrow$ after inflation ϕ and $\rho = V(\phi)$ exponentially smooth in space

good news: solved flatness, smoothness problems
bad news: we have done too much! too smooth!
can't form structures if density perfectly uniform

Quantum Mechanics to the Rescue

but quantum mechanics exists and is mandatory
classical ϕ field $\rightarrow$ quantized as inflatons
think $\vec{E}, \vec{B}$ vs photons

inflaton field must contain quantum fluctuations
before, during inflation

uncertainty principle: $\Delta x \Delta p \sim \hbar$
causal region at time t: Hubble length $\Delta x \sim d_H = c/H(t)$
expect momentum and energy fluctuations

$$c\Delta p \sim \Delta E \sim \hbar H \tag{5}$$

Q: implications?
Q: fate of fluctuations born a given scale $\lambda_{\rm init}$?
Q: analogy with Hawking radiation?

A Quantum Perturbation Factory

quantum mechanics: perturbations in energy $\rightarrow \delta\phi$
$\rightarrow$ different regions start inflation at different $V(\phi)$

`www: sketch of quantum perturbations during inflation`

quantum fluctuations born at scale $\lambda_{\rm init}$

- exponentially stretched until $\lambda > d_H$ "horizon crossing"
- then no longer causally connected
  $\rightarrow$ cannot "fluctuate back to zero"
- *"frozen in" as real density perturbations!*
  cosmic structures originate from quantum fluctuations!

Hawking radiation analogy:
uncertainty principle: $\Delta E \Delta t \sim \hbar$, so in timscale
$\Delta t \lesssim \hbar / m_\psi c^2$: particle pairs $\psi\bar{\psi}$ born and annihilate
black hole: one falls in, other emitted as thermal Hawking rad.
inflation: pair separated by expansion, "frozen" as fluctuation

Implications

If the inflationary model is true
density fluctuation "seeds" of cosmic structures are inflated quantum mechanical fluctuations

Q: how does this limit what we can know about them?

Q: what can *we hope to know?*

Q: what do we need to calculate?

Inflationary Fluctuations: What we need to know

quantum fluctuations are *random*
$\rightarrow$ impossible to predict locations, amplitudes of overdensities
$\rightarrow$ cannot predict location, mass, size of *any particular* cosmic object: galaxy/cluster/supercluster ...

but quantum mechanics does allow *statistical predictions*

What we want: *statistical* properties of fluctuations

- typical magnitude of fluctuations $\delta\phi$
- how $\delta\phi$ depends on lengthscales
- corresponding fluctuations in ρ_ϕ
- correlations at different length scales

Fluctuation Amplitude: Rough Estimate

quantum fluctuation $\rightarrow$ turn to uncertainty principle

$$\delta E \; \delta t \sim \hbar \sim 1 \tag{6}$$

recall: energy density is

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V \tag{7}$$

if perturbation from classical: $\phi(t,\vec{x}) = \phi_{\rm cl}(t) + \delta\phi(t,\vec{x})$,
then for small $\delta\phi \ll \phi_{\rm cl}$,

$$\delta\rho \sim (\nabla\delta\phi)^2 + V'(\phi_{\rm cl})\delta\phi \approx (\nabla\delta\phi)^2 \tag{8}$$

since slow roll $\rightarrow V'$ small (flat potential)

Q: what is characteristic volume for fluctuation?
Q: what is characteristic timescale δt?

H^{-1} is only lengthscale in problem
so $\nabla \delta\phi \sim \delta\phi / H^{-1} \Rightarrow \delta\rho \sim H^2 (\delta\phi)^2$
so in Hubble volume $V_H = d_H^3 = H^{-3}$, energy fluctuation is

$$\delta E = \delta\rho \ V_H = \frac{(\delta\phi)^2}{H} \tag{9}$$

characteristic timescale is $\delta t \sim 1/H$, so

$$\delta E \ \delta t \sim \frac{(\delta\phi)^2}{H^2} \sim 1 \tag{10}$$

and typical (root-mean-square) inflaton fluctuation is

$$\boxed{\delta\phi \sim H} \tag{11}$$

had to be! H is the only other
dimensionally correct scale in the problem!

Note: $H \sim const$ during inflation
all fluctuations created with $\sim$ same amplitude

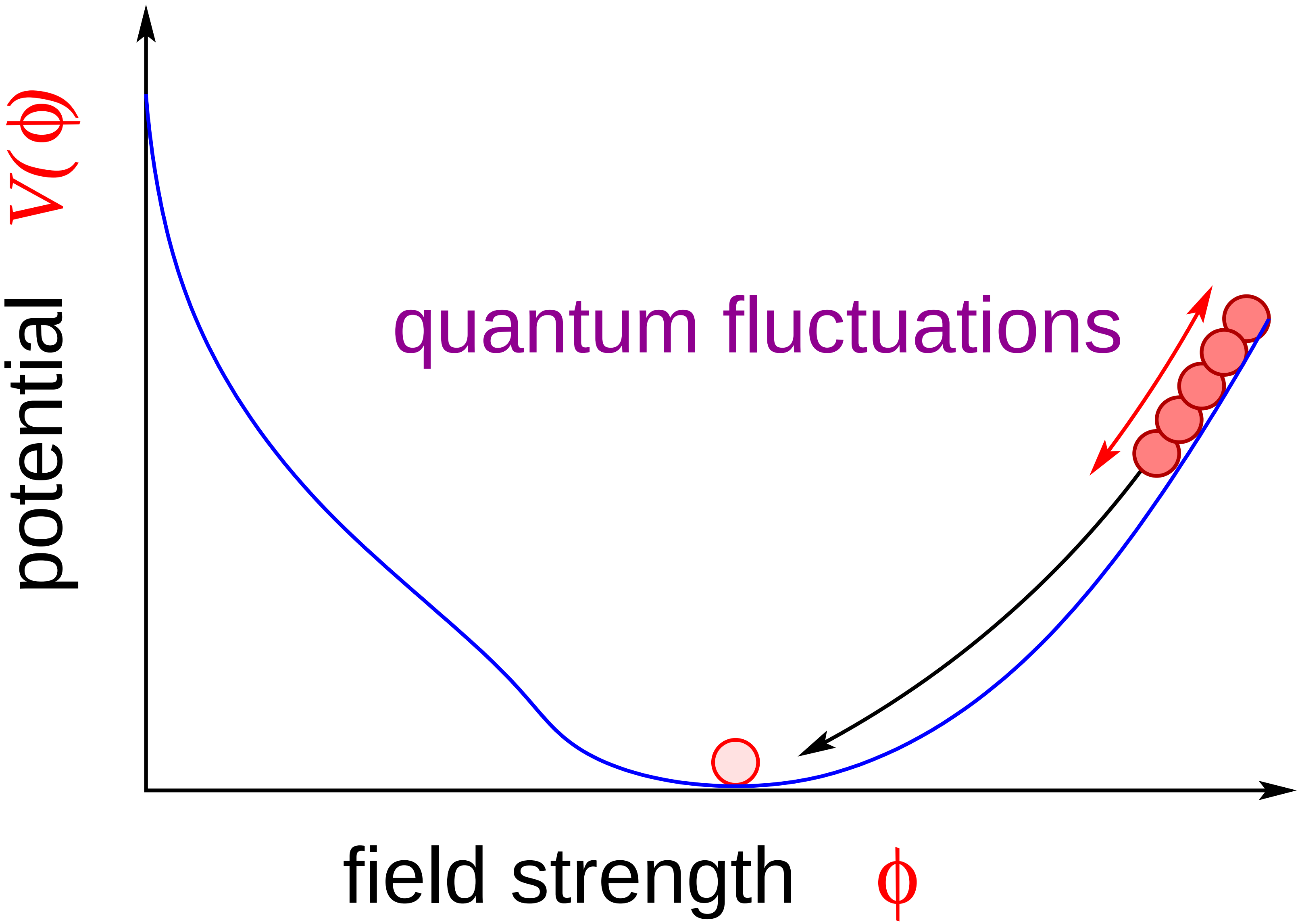
What Just Happened?

To summarize:

- *classically*, inflaton field ϕ_{cl} quickly inflates away any of its initial perturbations
- but *quantum fluctuations* $\delta\phi$ unavoidable created and persist throughout inflation
- in any region, amplitude $\delta\phi(\vec{x})$ random but *typical* value $\delta\phi \sim H$

Q: what do the presence of inflaton fluctuations mean for inflationary dynamics in different regions?

Q: what consequences/signatures of fluctuations might remain after inflation?


potential
V(ϕ)
quantum fluctuations
field strength
ϕ

Fluctuation Evolution and the Cosmic Horizon

in presence of fluctuations $\delta\phi$ and $\delta\rho_\phi$
can view inflationary universe as ensemble of "sub-universes"
evolving independently–same slow roll, but
with different ϕ, ρ_ϕ at a fixed t
classical discussion $\rightarrow$ ensemble average
now want behavior typical deviation from mean

particle horizon $\sim H^{-1}$ critical

- already saw: sets scale for fluctuation
- also "shuts off" fluctuation evolution

consider perturbation of lengthscale λ

- leaves horizon when $H \sim 1/\lambda$
- then can't evolve further: keeps same $\delta\rho / \langle\rho\rangle$
- until after inflation, when re-enters horizon

What Just Happened? ...Part Deux

the *classical* behavior of a slow-rolling ϕ
lead to homogeneity, isotropy
regardless of initial conditions
$\Rightarrow$ fixes horizon, flatness, monopole problem

the *quantum* fluctuations in ϕ
lead to density perturbations on all lengthscales
including scales $\gg d_{\text{hor}}$ today
these perturbations form the "seeds" for cosmic structures!

quantum mechanics & uncertainty principle
essential for the existence of cosmic structure

"The Universe is the ultimate free lunch."
– Alan Guth

Director's Cut Extras

Models for Inflation

Inflation model: specifies inflaton potential $V(\phi)$
[+ initial conditions, reheat prescription]

good news:
involves physics at extremely high energy scales
probed by observable signatures of inflation

bad news:
involves physics at extremely high energy scales
far beyond the reach of present-day or planned accelerators
no laboratory guidance or checks of inflationary physics

Q: possible physically reasonable choices for $V(\phi)$?

A Sample of Single-Field Potentials

Polynomial Potentials

e.g., Klein-Gordon $V = m^2\phi^2/2$, quartic $V = \lambda\phi^4$

- simplest models giving inflation
- require *Planck-scale* initial conditions for ϕ
- but to achieve sufficient inflation (enough e-foldings N) and perturbations at observed (CMB) scale demands *tiny coupling* $\lambda \sim 10^{-13}$ (!)
  $\rightarrow$ potential energy scale $V \ll m_{\rm pl}^4$
  but why is coupling so small?

Illustrates characteristics of (successful) inflation models:

▷ large initial $\phi \gtrsim m_{\rm pl}$ value

▷ small coupling in V $\rightarrow$ scale $V^{1/4} \sim 10^{15-16}$ GeV (GUT?)

Exponential Potentials: Power-Law Inflation

for potentials of the form

$$V = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{m_{\rm pl}}\right) \tag{12}$$

then can solve equations of motion exactly:

$$a \sim t^p \tag{13}$$

if $p > 1$, U. accelerates, but not exponentially

Designer Potentials

can customize V to give desired $a(t)$, e.g.,
to get $a \sim \exp(At^f)$, with $0 < f < 1$
then choose

$$V(\phi) \sim \left(\frac{\phi}{m_{\rm pl}}\right)^{-\beta}\left[1 - \frac{\beta^2}{6}\left(\frac{m_{\rm pl}^2}{\phi^2}\right)\right] \tag{14}$$

How about the Higgs?

from electroweak unification, we know of one scalar
$\rightarrow$ Higgs field H^0, $M_H \approx 125$ GeV

same symbol as Hubble, right kind of field $\rightarrow$ is it ϕ?
i.e., what about inflation at the electroweak scale?

not a bad idea–possibly correct!–but nontrivial at best
problem not with inflation, but its place in the cosmic dance

Amount of Inflation

during inflation scale factor grows exponentially
(in most models); in any case
quantify "amount" of inflation as
$N = \ln(a_{\rm fin}/a_{\rm init})$: number of "$e$-foldings"

What is needed?
to solve horizon, flatness, monopoles back to GUT scale:
$N \gtrsim N_{\rm min} \sim 60$ (PS6)

What is predicted?
Since $H = \dot{a}/a = d\ln a/dt = \dot{N}$, and $dt = d\phi/\dot{\phi}$, we have

$$N = \int_{t_{\rm init}}^{t_{\rm fin}} H\,dt = \int_{\phi_{\rm init}}^{\phi_{\rm fin}} \frac{H\,d\phi}{\dot{\phi}} \tag{15}$$

slow roll: $\dot{\phi} \simeq -V'/3H$, so

$$N = \int_{\phi_{\rm fin}}^{\phi_{\rm init}} \frac{3H^2\, d\phi}{V'} = m_{\rm pl}^2 \int_{\phi_{\rm fin}}^{\phi_{\rm init}} \frac{V}{V'} d\phi \tag{16}$$

typically expect $V'/V \sim 1/\phi$, which gives

$$N \sim \frac{\Delta\phi^2}{m_{\rm pl}^2} \tag{17}$$

amount of inflation set by:

- nature of potential V
- change in ϕ

note also that need $N \gg 1$ and thus
typically expect $\phi_{\rm init} \gtrsim m_{\rm pl}$
...but already required by slow roll

Q: what determines inflation end physically? mathematically?

Quantum Fluctuations: From ϕ to Density

at any given scale λ
relevant perturbation is the one born
during inflation when $\lambda \sim 1/H$

dimensionless perturbation amplitude:
fraction of mean density in horizon $\delta_H \equiv \delta\rho/\langle\rho\rangle$

on scale λ, amplitude fixed at horizon exit
$\delta\phi \sim H$ (in fact, $H/2\pi$)
$\rightarrow$ perturbed universe starts inflating at higher ϕ
or undergoes inflation for different duration $\delta t \simeq \delta\phi/\dot{\phi}$
this gives an additional expansion

$$\delta \ln a = \frac{\delta a}{a} = H\delta t = \frac{H^2}{2\pi\dot{\phi}} \tag{18}$$

but inflation exit is set at fixed ϕ_{end}
and fixed potential value $V_{\text{end}} \sim \rho_{\text{end}}$

perturbed energy density at end of inflation set by
different expansion at inflation exit:

$$\delta_{\text{H}} \equiv \frac{\delta\rho}{\langle\rho\rangle} \tag{19}$$

$$\sim \frac{\delta(a^3 V_{\text{end}})}{\left\langle a^3 V_{\text{end}} \right\rangle} \sim \frac{\delta a}{a} = \frac{H^2}{2\pi\dot{\phi}} \tag{20}$$

evaluated at any scale λ at horizon crossing
i.e., when $\lambda_{\text{com}} \sim 1/aH$
$\Rightarrow$ *density perturbations created at all lengthscales!*

caution: quick-n-dirty result
but gives right answer
in particular, fluctuation indep of lengthscale