Astro 507

Lecture 32

April 15, 2020

Announcements:

- **Preflight 6: Part (a) Due Friday April 17**
  Wikipedia Cosmology! propose a wikipedia upgrade
  can be modest and targeted, or more ambitious
  FWIW: I have a animation suggestion

Last time: inflation and the homogeneous universe

- a graceful exit from inflation *Q: how*
- inflation in the order of cosmic epoch *Q: namely?*

Today: inflation and inhomogeneities, and observational tests

- spoiler: we are all quantum fluctuations!
- overview/concepts in class–big picture
  (long) technical discussion in Extras

Classical Inflation and Smoothness

expect initial spatial inhomogeneities in $\phi(\vec{x})$
but evolves classically as

$$\ddot{\phi} - \nabla^2\phi + 3H\dot{\phi} - V' = 0 \tag{1}$$

where

$$\nabla^2 = \sum \frac{\partial^2}{\partial x^2_{\text{phys}}} = \frac{1}{a^2} \sum \frac{\partial^2}{\partial x^2_{\text{com}}} \tag{2}$$

inhomogeneities $\delta\phi(\vec{x})$ measured by nonzero gradients
but since $\nabla^2 \propto 1/a^2 \to 0$ exponentially, classically: $\delta\phi(\vec{x}) \to 0$
$\Rightarrow$ after inflation ϕ and $\rho = V(\phi)$ exponentially smooth in space

good news: solved flatness, smoothness problems
bad news: we have done too much! too smooth!
can't form structures if density perfectly uniform

Quantum Mechanics to the Rescue

but quantum mechanics exists and is mandatory
classical ϕ field $\rightarrow$ quantized as inflatons
think $\vec{E}, \vec{B}$ vs photons

inflaton field must contain quantum fluctuations
before, during inflation

uncertainty principle: $\Delta x \Delta p \sim \hbar$
causal region at time t: Hubble length $\Delta x \sim d_H = c/H(t)$
expect momentum and energy fluctuations

$$c\Delta p \sim \Delta E \sim \hbar H \tag{3}$$

Q: implications?
Q: fate of fluctuations born a given scale λ_{init}?
Q: analogy with Hawking radiation?

A Quantum Perturbation Factory

quantum mechanics: perturbations in energy $\rightarrow \delta\phi$
$\rightarrow$ different regions start inflation at different $V(\phi)$

`www: sketch of quantum perturbations during inflation`

quantum fluctuations born at scale $\lambda_{\rm init}$
- exponentially stretched until $\lambda > d_H$ "horizon crossing"
- then no longer causally connected
  $\rightarrow$ cannot "fluctuate back to zero"
- *"frozen in" as real density perturbations!*
  cosmic structures originate from quantum fluctuations!

Hawking radiation analogy:
uncertainty principle: $\Delta E \Delta t \sim \hbar$, so in timescale
$\Delta t \lesssim \hbar/m_\psi c^2$: particle pairs $\psi\bar{\psi}$ born and annihilate
black hole: one falls in, other emitted as thermal Hawking rad.
inflation: pair separated by expansion, "frozen" as fluctuation

Implications

If the inflationary model is true
density fluctuation “seeds” of cosmic structures are inflated quantum mechanical fluctuations

Q: how does this limit what we can know about them?

Q: what can we hope to know?

Q: what do we need to calculate?

Inflationary Fluctuations: What we need to know

quantum fluctuations are *random*
$\rightarrow$ impossible to predict locations, amplitudes of overdensities
$\rightarrow$ cannot predict location, mass, size of *any particular* cosmic object: galaxy/cluster/supercluster ...

but quantum mechanics does allow *statistical predictions*

What we want: *statistical* properties of fluctuations

- typical magnitude of fluctuations $\delta\phi$
- how $\delta\phi$ depends on lengthscales
- corresponding fluctuations in ρ_ϕ
- correlations at different length scales

Fluctuation Amplitude: Rough Estimate

quantum fluctuation $\rightarrow$ turn to uncertainty principle

$$\delta E \ \delta t \sim \hbar \sim 1 \tag{4}$$

recall: energy density is

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V \tag{5}$$

if perturbation from classical: $\phi(t,\vec{x}) = \phi_{\rm cl}(t) + \delta\phi(t,\vec{x})$, then for small $\delta\phi \ll \phi_{\rm cl}$,

$$\delta\rho \sim (\nabla\delta\phi)^2 + V'(\phi_{\rm cl})\delta\phi \approx (\nabla\delta\phi)^2 \tag{6}$$

since slow roll $\rightarrow$ V' small (flat potential)

Q: what is characteristic volume for fluctuation?
Q: what is characteristic timescale δt?

H^{-1} is only lengthscale in problem
so $\nabla\delta\phi \sim \delta\phi/H^{-1} \Rightarrow \delta\rho \sim H^2(\delta\phi)^2$
so in Hubble volume $V_H = d_H^3 = H^{-3}$, energy fluctuation is

$$\delta E = \delta\rho\ V_H = \frac{(\delta\phi)^2}{H} \tag{7}$$

characteristic timescale is $\delta t \sim 1/H$, so

$$\delta E\ \delta t \sim \frac{(\delta\phi)^2}{H^2} \sim \hbar \tag{8}$$

and typical (root-mean-square) inflaton fluctuation is

$$\boxed{\delta\phi \sim \hbar\, H} \tag{9}$$

had to be! H is the only other
dimensionally correct scale in the problem!

∞ Note: $H \sim const$ during inflation
all fluctuations created with $\sim$ same amplitude

Intermission: Questions?

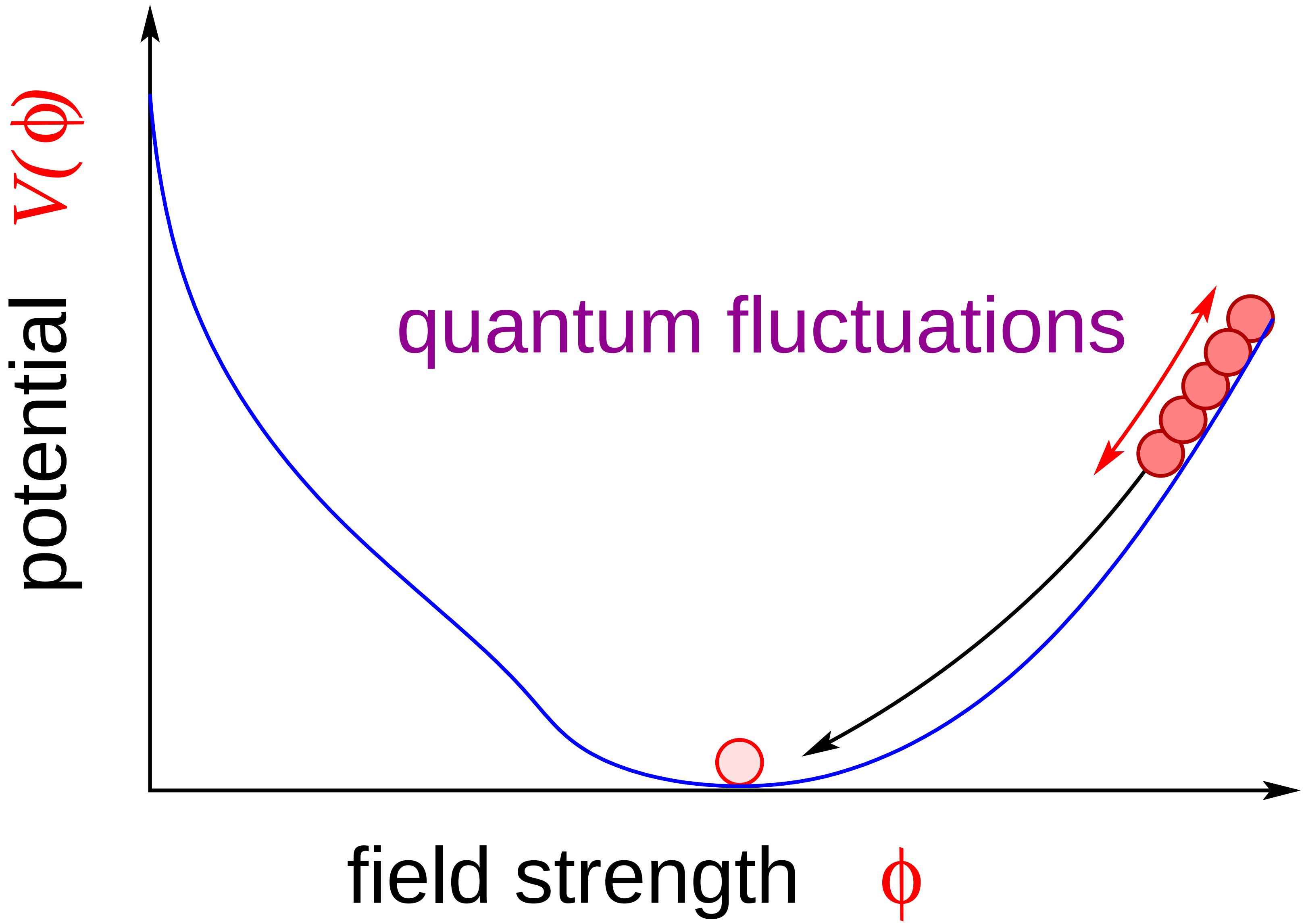
What Just Happened?

To summarize:

- *classically*, inflaton field ϕ_{cl} quickly inflates away any of its initial perturbations
- but *quantum fluctuations* $\delta\phi$ unavoidable created and persist throughout inflation
- in any region, amplitude $\delta\phi(\vec{x})$ random but *typical* value $\delta\phi \sim H$

Q: what do the presence of inflaton fluctuations mean for inflationary dynamics in different regions?

Q: what consequences/signatures of fluctuations might remain after inflation?


potential $V(\phi)$
quantum fluctuations
field strength ϕ

Fluctuation Evolution and the Cosmic Horizon

in presence of fluctuations $\delta\phi$ and $\delta\rho_\phi$
can view inflationary universe as ensemble of "sub-universes"
evolving independently–same slow roll, but
with different ϕ, ρ_ϕ at a fixed t
classical discussion $\rightarrow$ ensemble average
now want behavior typical deviation from mean

particle horizon $\sim H^{-1}$ critical

- already saw: sets scale for fluctuation
- also "shuts off" fluctuation evolution

consider perturbation of lengthscale λ

- leaves horizon when $H \sim 1/\lambda$
- then can't evolve further: keeps same $\delta\rho/\langle\rho\rangle$
- until after inflation, when re-enters horizon

What Just Happened? ...Part Deux

the *classical* behavior of a slow-rolling ϕ
lead to homogeneity, isotropy
regardless of initial conditions
$\Rightarrow$ fixes horizon, flatness, monopole problem

the *quantum* fluctuations in ϕ
lead to density perturbations on all lengthscales
including scales $\gg d_{\text{hor}}$ today
these perturbations form the "seeds" for cosmic structures!

quantum mechanics & uncertainty principle
essential for the existence of cosmic structure
"The Universe is the ultimate free lunch."
– Alan Guth

The Cosmic Harmonic Oscillator

in Director's Cut notes:

- inflaton field begins in vacuum state
- quantum fluctuations evolve as a *simple harmonic oscillator*

$\rightarrow$ dominated by *vacuum=ground state*

Q: wavefunction of ground state simple harmonic oscillator?
Q: probability of finding particle at x*?*

Q: implications for inflaton fluctuations?

Inflation Spectrum
Statistical Properties

$\star$ Recall: inflaton quantum modes $\leftrightarrow$ harmonic oscillator
dominated by vacuum $\leftrightarrow$ ground state $\|\psi_{\text{sho}}(x)\|^2 \sim e^{-x^2/2\Delta x^2}$
$\phi_k \leftrightarrow x$ fluctuations are statistically Gaussian
i.e., perturbations of all sizes occur, but
probability of finding perturbation of size $\delta(R)$
on scale R is distributed as a Gaussian

$\star$ inflaton perturbations $\rightarrow$ reheating
$\rightarrow$ radiation, matter perturbations
same levels in both: "adiabatic"

$\star\star\star\star\star$ All of these are bona fide *pre*dictions of inflation
and are testable! *Q: how?*

Inflation Spectrum
Slightly Tilted Scale Invariance

recall: perturbation leaving horizon have very similar amplitude during inflation $\rightarrow$ nearly same for all lengthscales $\leftrightarrow k$ perturbation rms amplitude

$$\delta^2_{\mathrm{inf}}(k) \propto k^{-6\epsilon+2\eta} \qquad (10)$$

⋆ successful inflation $\Leftrightarrow$ slow roll $\Leftrightarrow \epsilon, \eta \ll 1$ demands
perturbation spectrum nearly independent of scale
nearly "self-similar," without characteristic scale
"Peebles-Harrison-Zel'dovich" spectrum

⋆ successful inflation must end $\rightarrow \epsilon, \eta \neq 0$
demands small departures from scale-invariance
"tilted spectrum"

Inflation Creates Primordial Gravity Waves

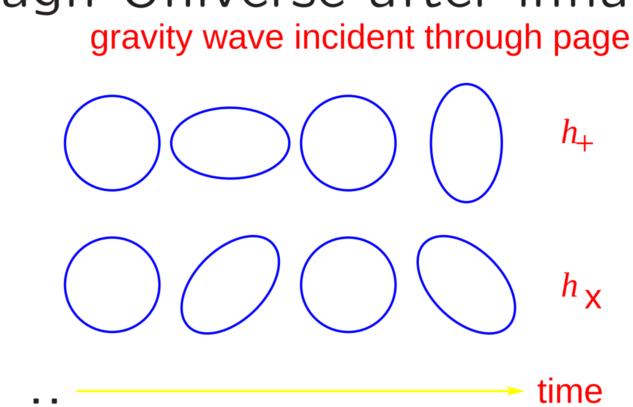
Inflaton field fluctuations are inhomogeneous perturbations
to cosmic mass-energy density field

can excite **gravitational radiation**

when fluctuations have nonzero quadrupole, i.e., **tensor modes**

- **cosmic gravitational wave background**
- wavelengths span all scales up to Mpc
- wave amplitude directly related to density perturbations
- waves propagate unimpeded through Universe after inflation

effect on ring of test particles



Q: how to test?

Searching for Primordial Gravitational Waves

- waves drive quadrupole motion
  introduce **CMB polarization**
  we'll see: gravitational wave excite B modes–curl features

- In principle: direct detection possible via spacetime effects!
  but cosmo signal below astro events (BH, NS)
  not accessible to aLIGO/VIRGO, likely not LISA.
  `www: gravitational wave signal comparison`

Testing Inflation: Status to Date

test inflation by measuring density fluctuations
and their statistical properties
on various scales at various epochs

CMB at large angles (large scales, decoupling)

- nearly scale invariant! woo hoo! (COBE 93)
- Gaussian distribution (COBE, WMAP, Planck)
  `www: 3-yr WMAP` T `distribution`
- WMAP, Planck: evidence for tilt! favors large scales ("red")!
  Planck (2013): $\alpha = -0.035 \pm 0.004$ nonzero at $\sim 9\sigma$!

These did not have to be true!
Not guaranteed to be due to inflation
but very encouraging nonetheless

Inflation Scorecard

Summary:
Inflation designed to solve horizon, flatness, smoothness
does this, via accelerated expansion driven by inflaton

But unexpected bonus: structure
inflaton field has quantum fluctuations
imprinted before horizon crossing
later return as density fluctuations
$\rightarrow$ inflationary seeds of cosmic structure?!

Thus far: observed cosmic density fields
have spectrum, statistics as predicted by inflation

The frontier: CMB polarization probes of cosmic gravity waves

Director's Cut Extras

Fluctuation Spectrum: In More Detail

Starting point of more rigorous treatment
in equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + V'(\phi) = 0 \tag{11}$$

write field as sum

$$\phi = \phi_{\text{classical}}(t) + \delta\phi(t, \vec{x}) \tag{12}$$

- **classical amplitude** $\phi_{\text{cl}}(t)$
  spatially homogeneous: *smooth, classical, background* field
  evolves according to classical equation of motion
  $\rightarrow$ this has been our focus thus far; now add
- **quantum fluctuations** $\delta\phi(t, \vec{x})$
  these can vary across space and with time

decompose spatial part of fluctuations into plane waves

$$\delta\phi(t,\vec{x}) = \sum_{\vec{k}} \delta\phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}_{\rm com}} \tag{13}$$

convenient to label Fourier modes by
comoving wavelength $\lambda \equiv \lambda_{\rm com}$, wavenumber $k \equiv k_{\rm com} = 2\pi/\lambda_{\rm com}$
but *physical* wavelength $\lambda_{\rm phys} = a\lambda_{\rm com}$, wavenumber $k_{\rm phys} = k/a$

as long as quantum perturbations $\delta\phi$ small (linear evolution)
each wavelength–i.e., scale–evolves independently
$\rightarrow$ main reason to use Fourier modes

classically $\delta\phi = (\delta\phi)^2 = 0$ by definition!
Q: what is physical significance of quantum excitations in ϕ?

Slow Roll and Scale Dependence

Last time, and in Extras today:
dimensionless fluctuation amplitude (variance)
at comoving wavenumber $k = k_{\rm com}$

$$\Delta^2(k) \sim \left(\frac{\delta\rho}{\rho}\right)_k \sim \left(\frac{H^2}{m_{\rm pl}^2}\right)\left(\frac{H}{\dot\phi}\right)^2_{aH=k} \sim \left(\frac{V}{\epsilon m_{\rm pl}^4}\right)_{aH=k} \tag{14}$$

evaluated at "horizon crossing" $aH = k$

Q: how does aH change during inflation?
Q: for slow roll, how does $\Delta^2(k)$ change with scale?

The Quantum Inflaton Field

quantum mechanically:

- true ϕ has fluctuations around background value
- each $\vec{k}$ mode $\leftrightarrow$ independent quantum states (oscillators)
- mode fluctuations *quantized* $\rightarrow$ quanta are inflaton particles analogous to photons as EM quanta
- occupation numbers: $n_{\vec{k}} > 0 \rightarrow$ real particles present
- if $n_{\vec{k}} = 0 \rightarrow \langle \delta\phi \rangle = 0$ no particles (vacuum/ground state) but zero-point fluctuations still present $\left\langle \delta\phi^2 \right\rangle \neq 0$

Fluctuation Lagrangian

expand each $\vec{k}$ mode around classical value

$$\begin{aligned} \mathcal{L}_{\vec{k}} &= \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 - \frac{1}{2}V''(\phi_{\text{cl}})\delta\phi_{\vec{k}}^2 - V(\phi_{\text{cl}}) && (15) \\ &\approx \frac{1}{2}\delta\dot{\phi}_{\vec{k}}^2 - \frac{1}{2}\frac{k^2}{a^2}\delta\phi_{\vec{k}}^2 && (16) \end{aligned}$$

where slow roll $\rightarrow$ potential terms small
$\rightarrow$ a **massless simple harmonic oscillator**

$\delta\phi$ vacuum state: zero point fluctuations
formally same a quantum harmonic oscillator!
for *each k mode*, fluctuation *amplitudes random*
but probability distribution is like $n = 0$ oscillator

$$P(\delta\phi_{\vec{k}}) \propto e^{-\delta\phi_{\vec{k}}^2/2\sigma_{\vec{k}}^2} \qquad (17)$$

where variance $\sigma_{\vec{k}}^2 = \left\langle \delta\phi_{\vec{k}}^2 \right\rangle$
$\rightarrow$ vacuum fluctuation amplitudes have *gaussian* distribution

Total ϕ energy density is $\rho_\phi = \rho_{\text{background}} + \rho_{\text{zeropoint}} + \rho_{\text{particles}}$
pre-inflation: could have $\rho_{\text{particles}} \neq 0$
in fact: if thermalized, $\rho_{\text{particles}} \propto T^4$ (radiation)
$\rightarrow$ inflation only begins when $\rho_{\text{background}} \gg \rho_{\text{particles}}$
Q: what happens to inflatons after inflation begins?

after inflation begins, universe rapidly expanded, cooled
inflatons diluted away
$\rightarrow$ inflation field driven to **vacuum (ground) state**

Since ϕ in quantum vacuum state: fluctuations are zero-point
$\rightarrow$ **gaussian distribution** of amplitudes in each k mode
$\rightarrow$ strong prediction of slow-roll inflation

now want to solve for size of rms σ_k at each mode

classically, perturbations have equation of motion

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi + V''\delta\phi = 0 \tag{18}$$

$$\frac{d^2}{dt^2}\delta\phi + 3H\frac{d}{dt}\delta\phi + \frac{k^2}{a^2}\delta\phi \approx 0 \tag{19}$$

(in slow roll: V'' term negligible)

Sketch of Quantum Treatment

Promote $\delta\phi \to$ operator $\widehat{\delta\phi}$
plane wave expansion: $\widehat{\delta\phi} = \sum_{\vec{k}} \widehat{\delta\phi}_{\vec{k}}$
introduce annihilation, creation operators $\hat{a}_{\vec{k}}$, $\hat{a}^{\dagger}_{-\vec{k}}$, then

$$\widehat{\delta\phi}_{\vec{k}} = w_k(t)\,\hat{a}_{\vec{k}} + w_k^*(t)\,\hat{a}^{\dagger}_{-\vec{k}} \tag{20}$$

where $w_k(t)$ is a solution of field equation

$$\ddot{w}_k + 3H\dot{w}_k + \left(\frac{k}{a}\right)^2 w_k = 0 \tag{21}$$

Compare limits:

- $k/a \gg H \to k \gg aH \to \lambda \ll 2\pi d_{H,\mathrm{com}}$
  Q: physical interpretation of limit?
  w_k evolves as harmonic oscillator (free massless field)
- $k/a \ll H \to k \ll aH \to \lambda \gg 2\pi d_{H,\mathrm{com}}$
  Q: physical interpretation of limit?
  $\dot{w}_k \propto a^{-3} \to 0 \to w_k$ value "frozen"

Inflation Perturbations: Evolution and Horizons

sub-horizon scales $\lambda \ll 2\pi d_{H,\mathrm{com}}$
inflaton fluctuations $\delta\phi$ are causally connected
evolve like harmonic oscillator $\rightarrow$ rms amplitude $\left\langle |w_k|^2 \right\rangle$ *constant*

but cosmic acceleration during inflation $\rightarrow$ $d_{H,\mathrm{com}}$ *shrinks*
since $\dot{d}_{H,\mathrm{com}} = d(aH)^{-1}/dt = d(\dot{a}^{-1})/dt = -\ddot{a}/\dot{a}^2 < 0$ during inf
$d_{H,\mathrm{com}}$ shrinkage: initially sub-horizon scales $\rightarrow$ super-horizon

super-horizon scales $\lambda \gg 2\pi d_{H,\mathrm{com}}$
fluctuations out of causal contact
amplitude "frozen in" until post-inflation $d_{H,\mathrm{com}}$ regrows

Inflation Perturbations: Spectrum of Amplitudes

examine fluctuations from vacuum
$\rightarrow$ find expected amplitudes w_k

since fluctuations have *quantum* origin
- cannot predict definite values for mode amplitudes, phases
- but *can* predict statistical properties

for *different* modes $\vec{k}$ and $\vec{k}'$,
Q: what do we expect?

for *different* modes $\vec{k}$ and $\vec{k}'$,
expectation is

$$\langle\widehat{\delta\phi}_{\vec{k}}\widehat{\delta\phi}_{\vec{k}'}\rangle = w_k w_{k'}\left\langle\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}'}\right\rangle + \text{c.c.} = 0 \tag{22}$$

because $\left\langle\hat{a}_{\vec{k}}\hat{a}^\dagger_{\vec{k}'}\right\rangle = \left\langle\hat{a}_{\vec{k}}\right\rangle\left\langle\hat{a}^\dagger_{\vec{k}'}\right\rangle = 0$
$\Rightarrow$ modes are statistically independent
note: true even if $|\vec{k}| = |\vec{k}'| = k$ but $\vec{k}\cdot\vec{k}' = 0$
i.e., different directions $\vec{k} = k\hat{x}, \vec{k}' = k\hat{y}$
$\Rightarrow$ phase $e^{i\vec{k}\cdot\vec{x}}$ is random

for a single mode k, vacuum expectation is

$$\langle\widehat{\delta\phi}^2_{\vec{k}}\rangle = |w_k|^2\left\langle\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right\rangle = |w_k|^2 \neq 0 \tag{23}$$

$$= \frac{H^2}{2L^3k^3} \tag{24}$$

where last expression
- from full quantum calculation, in box of size L
- to be evaluated at horizon crossing: $k_{\text{phys}} = H \to k = aH$

each in phase space volume

$$d^3xd^3k = \frac{1}{(2\pi L)^3}4\pi k^2dk = \frac{4\pi k^3}{(2\pi L)^3}\frac{dk}{k} \tag{25}$$

then fluctuation amplitude is

$$P_\phi(k)\frac{dk}{k} \equiv \frac{4\pi k^3}{(2\pi L)^3}|\delta\phi_k|^2\frac{dk}{k} = \left(\frac{H}{2\pi}\right)^2\frac{dk}{k} \tag{26}$$

and so the phase space fluctuation density in ϕ is

$$P_\phi(k) = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{27}$$

as before, but now

- explicitly seen independence of k
- know when to evaluate: at horizon crossing $k = aH$

Fluctuation Evolution and Spectrum

consider some fixed (comoving) scale $\lambda = 2\pi/k$ key idea: causal physics acts until $\lambda > d_{\mathsf{H,com}}$: "horizon crossing"
$\rightarrow$ quantum fluctuations laid down while inside $d_{\mathsf{H,com}}$
"frozen in" once outside of $d_{\mathsf{H,com}}$

from last time: quantum analysis gives fluctuation variance

$$\left\langle (\delta\phi_k)^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2_{k=aH} \tag{28}$$

to be evaluated at horizon crossing: $k = 1/d_{\mathsf{H,com}} = aH$

Fluctuation Evolution and Spectrum

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to be evaluated at horizon crossing: $k = 1/d_{\mathrm{H,com}} = aH$

Inflationary Density Perturbations: Spectrum

Recall: density fluctuations $\rightarrow$ start inflating earlier (later)
$\rightarrow$ more (less) expansion than average
extra scale factor boost $\delta a/a = H\delta t = H\delta\phi/\dot{\phi}$ $\rightarrow$ larger volume
$\rightarrow$ density perturbations have mean square

$$\delta^2_{\mathrm{inf}}(k) \equiv \left(\frac{\delta\rho}{\rho}\right)^2_k \tag{30}$$

$$\sim \left(\frac{\delta a}{a}\right)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 (\delta\phi)^2 = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \tag{31}$$

evaluated at $aH = k$

slow roll: H, $\dot{\phi}$ slowly varying
$\rightarrow$ expect fluctuation amplitude $\sim H^4/\dot{\phi}^2 \sim const$
over wide range of k

In particular: slow roll $\dot{\phi} = -3V'/H$,
and $H^2 = V/3m_{\rm pl}^2$, which gives

$$\delta_{\rm inf}^2(k) = \frac{1}{12\pi^2 m_{\rm pl}^6}\left(\frac{V^3}{V'^2}\right) = \frac{1}{24\pi^2 m_{\rm pl}^4}\left(\frac{V}{\epsilon}\right) \tag{32}$$

where $\epsilon = m_{\rm pl}(V'/V)^2/2$

anticipating $\sim$ power law behavior,
define $\delta_{\rm inf}^2(k) \sim k^{\alpha(k)}$
then scale dependence is

$$\alpha(k) = \frac{d\ln\delta_{\rm inf}^2(k)}{d\ln k} \tag{33}$$

evaluated when comoving scale $k = aH$ crosses horizon
i.e., this relates k to homogeneous a, ϕ values

Underlying physical question:
how do cosmic properties–e.g., $H, \rho \approx V$–*change* while the universe inflates as it slowly rolls?

- if no change $\rightarrow \dot{\phi} = 0 \rightarrow$ same V, H always $\rightarrow \epsilon = 0$
  all scales see same U when leaving horizon $k = aH$
  $\rightarrow$ all scales have same quantum fluctuations
- but *slow* roll $\neq$ *no* roll!
  $\dot{\phi} \neq 0 \rightarrow$ U properties *do* change

recall: $\delta^2_{\text{inf}}(k) \propto V/\epsilon$
and as slowly roll $\rightarrow V$ decreasing, ϵ increasing
and horizon scale $d_{H,\text{com}}$ also decreases
Q: so which scales get larger perturbations? smaller?

because V decreasing, ϵ increasing

$\delta^2_{\mathrm{inf}}(k) \propto V/\epsilon$ decreases with time

$\rightarrow$ smaller perturbations later in slow roll

since horizon scale $d_{H,\mathrm{com}}$ decreases

later times $\leftrightarrow$ smaller scales

$\Rightarrow$ slow roll $\rightarrow$ *smaller* perturbations on *smaller* scales

$\Rightarrow$ perturbation spectrum *tilted* to large scales $\rightarrow$ small k

in slow roll, $k = aH$ change mostly due to a:

$$d \ln k \approx d \ln a = \frac{da}{a} = H \; dt \tag{34}$$

recast in terms of inflaton potential

$$= \frac{H d\phi}{\dot{\phi}} = -3 \frac{H^2}{V'} d\phi \tag{35}$$

and so

$$\frac{d}{d \ln k} = -m_{\mathrm{pl}}^2 \frac{V'}{V} \frac{d}{d\phi} \tag{36}$$

Using this, can show:

$$\alpha(k) = \frac{d \ln \delta_{\mathrm{inf}}^2(k)}{d \ln k} = -6\epsilon + 2\eta \tag{37}$$

i.e., perturbation spectrum $\delta_{\mathrm{inf}}^2(k) \propto k^{-6\epsilon+2\eta}$

Major result!

Q: why? what does this mean physically? for cosmology? for inflation?

Gravity Waves: Tensor Perturbations

⋆ so far: only looked at density (scalar) perturbations
but also tensor perturbations → gravity waves!

what's really going on: *cosmic metric* is perturbed
spatial part (in a particular coordinate system = gauge):
• unperturbed = FLRW

$$d\ell^2|_{\mathrm{FLRW}} = a(t)^2\ (dx^2 + dy^2 + dz^2) = a(t)^2\ \delta_{ij}\ dx_i\ dx_j \quad (38)$$

with perturbations

$$d\ell^2|_{\mathrm{pert}} = a(t)^2\ e^{2\zeta}\ \gamma_{ij}\ dx_i\ dx_j \quad (39)$$

with *curvature perturbation* the *scalar* function $\zeta(\vec{x}, t)$
Q: what it its physical effect?

perturbed metric

$$d\ell^2|_{\text{pert}} = a(t)^2 \; e^{2\zeta} \; \gamma_{ij} \; dx_i \; dx_j \tag{40}$$

curvature perturbation scalar function $\zeta(\vec{x}, t)$ changes local volume

$\rightarrow$ locally: isotropic stretching

tensor perturbation is, to lowest order

$$\gamma_{ij} \approx \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ -h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij} + \begin{pmatrix} h_+ & h_\times & 0 \\ -h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{41}$$

with *two independent modes* of amplitude $h_+, h_\times$

Q: physical effect of these modes?

tensor perturbation is, to lowest order

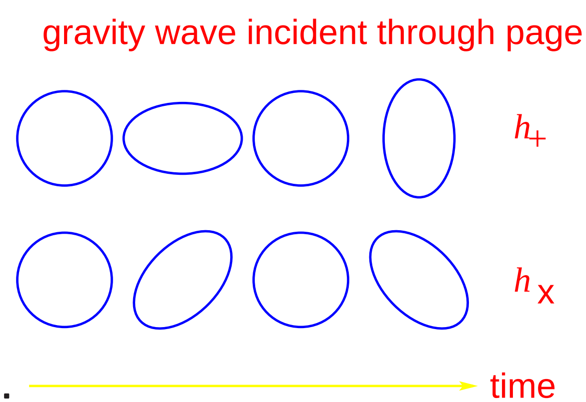
$$\gamma_{ij} \approx \delta_{ij} + \begin{pmatrix} h_+ & h_\times & 0 \\ -h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{42}$$

looks like rotation: roughly speaking preserves volume but changes angles

moreover: h satisfies massless wave equation!

$h \Leftrightarrow$ **gravitational radiation**

effect on a ring of test particles:



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Metric Fluctuations

tensor perturbations directly are metric perturbation
what about the inflaton perturbations?

curvature perturbation in an invariant (coordinate independent):

$$\zeta = \Phi + H\delta t = \Phi + H\frac{\delta\phi}{\dot{\phi}} \tag{43}$$

$\Phi(\vec{x}, t)$ is local *gravitational potential perturbation*

inflation fluctuations ϕ also are metric perturbations
but amplitude differs from gravity wave amplitude
by factor $H/\dot{\phi}$
and thus scalar perturbation variance differs by factor

$$r = \frac{\Delta_h^2}{\Delta_\Phi^2} \sim \left(\frac{\dot{\phi}}{H}\right)^2 \sim \epsilon \tag{44}$$

Inflationary Tensor Perturbations

variance as a function of scale (wavenumber)

$$\Delta_h^2(k) \sim \left(\frac{V}{m_{\mathrm{pl}}^4} \right)_{aH=k} \tag{45}$$

- evaluated at "horizon crossing" $aH = k$
- **directly probes inflation potential $V(\phi)$!**
- compare to density ("scalar") perturbations:
  tensor-to-scalar ratio

$$r = \frac{\Delta_h^2}{\Delta_\Phi^2} = 16\epsilon \tag{46}$$

- for $\epsilon \ll 1$, expect $r \ll 1$: scalar dominates