Astro 507
Lecture 35
April 22, 2020

Announcements:

- **Problem Set 6** **extended Monday April 27**
  after this: final Problem Set due Finals Week
  recall: lowest PF and PS dropped
  can drop Finals Week PS

Last time:

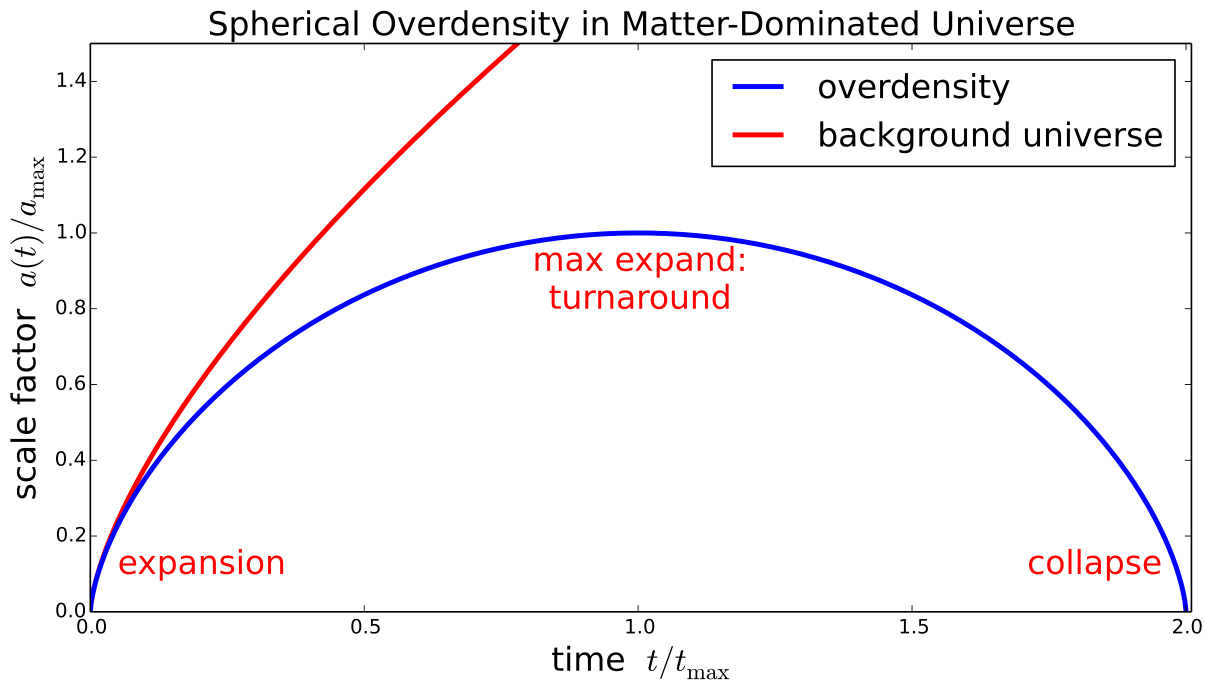
spherical collapse model

Q: what's that? assumptions?

Q: behavior? lessons? why useful?

quantifying density perturbations

- many ways to do this: an ongoing open question
- given (smoothed) density field $\rho(\vec{x})$
  useful to define *density contrast* $\delta(\vec{x})$
  Q: what's that?



- initially expand with Universe
- but extra gravity in overdensity slows expansion
- reach max expansion at $t_{\rm max}$, then begin collapse
  "turnaround" epoch
- formally, collapse (to a point!) at $t_{\rm coll} = 2t_{\rm max}$
- in reality: after turnaround, infalling matter virializes
  marks birth of halo as collapsed object

Quantifying Density Fluctuations

Given $\rho(t, \vec{x})$, define

mean (average) density $\langle \rho \rangle = \langle \rho(t, \vec{x}) \rangle = \rho_{\mathrm{FRW}}(t)$: "background"
(suppress t hereafter)

density fluctuation $\delta\rho(\vec{x}) = \rho(\vec{x}) - \langle \rho \rangle$

density contrast

$$\delta(\vec{x}) = \frac{\delta\rho}{\rho} = \frac{\rho(\vec{x}) - \langle \rho \rangle}{\langle \rho \rangle} \tag{1}$$

where $\delta \neq \delta_{\mathrm{Dirac}}$!

Q: possible range of δ values?

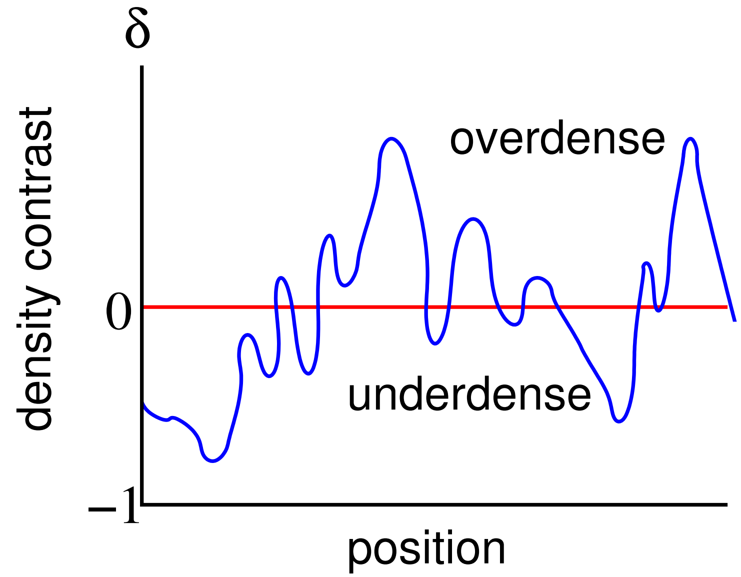
Q: what is $\langle \delta \rangle$?

Q: how does cosmic expansion affect δ?

key measure of cosmic structure:
density contrast

$$\delta(\vec{x}) \;=\; \frac{\delta\rho}{\rho} \equiv \frac{\rho(\vec{x}) - \langle\rho\rangle}{\langle\rho\rangle} \in (-1, \infty)$$

where average is over large volume V

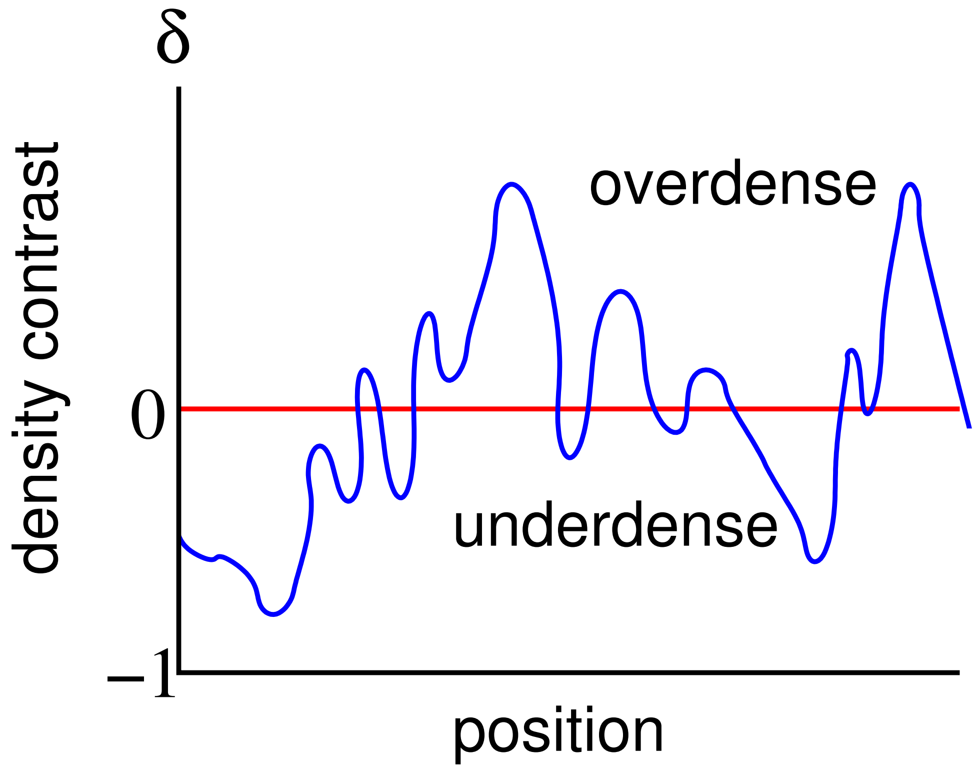


Q: what is the order-of-magnitude of the density contrast in this room? of the Galactic ISM?

by definition: $\langle\delta\rangle = \frac{1}{V}\int d^3x\,\delta(\vec{x}) = 0$

would like to study structures on different cosmic lengthscales λ
Q: how to do this using density contrast?

Density is Destiny: Inhomogeneous Version



Q: given these initial conditions, what happens?

Q: where will be first collapsed objects? first voids?

Q: what δ needed to have a bound structure?

Spherical Collapse Revisited

in spherical collapse model we can calculate overdensity:
since "subuniverse" has $\rho \propto 1/a^3$

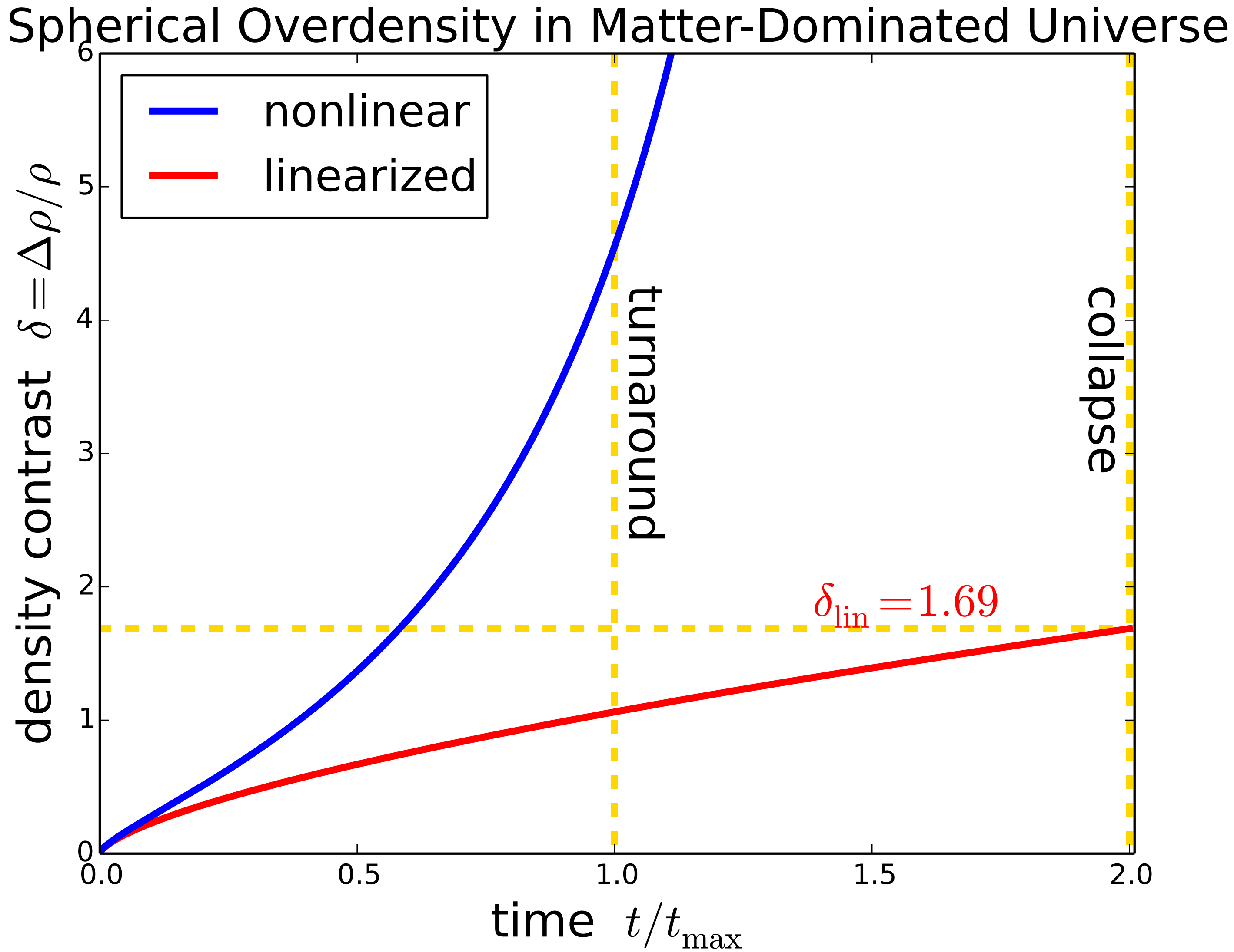
$$\delta(t) = \frac{\rho(t)}{\rho_{\rm bg}(t)} - 1 = \left(\frac{a_{\rm bg}}{a}\right)^3 - 1 \tag{2}$$

with $a_{\rm bg} \propto t^{2/3}$ the matter-dom background
$\rightarrow$ exact nonlinear solution (pre-virial)

For small t, to first order $a(t) \sim t^{2/3} = a_{\rm bg}(t)$:
recover background result; $\delta(t) = 0$
PS6–to second order: $a(t) = a_{\rm bg}(t)[1 - (12\pi t/t_{\rm coll})^{2/3}/20]$

$$\delta(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\rm coll}}\right)^{2/3} = \delta_{\rm lin}(t) \tag{3}$$

"linearized" density contrast: $\delta_{\rm lin}(t) \propto t^{2/3} \propto a_{\rm bg}$


Spherical Overdensity in Matter-Dominated Universe
nonlinear
linearized
turnaround
collapse
$\delta_{\rm lin} = 1.69$
density contrast $\delta = \Delta\rho/\rho$
time $t/t_{\rm max}$
0
1
2
3
4
5
6
0.0
0.5
1.0
1.5
2.0

Very useful result:

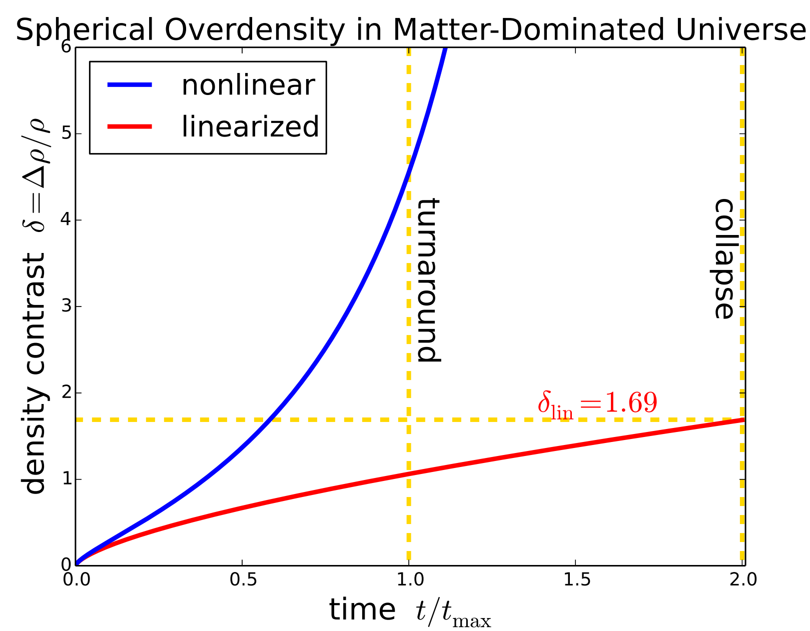
$$\delta_{\text{nonlin}}(t) = \left(\frac{a_{\text{bg}}}{a_{\text{nonlin}}}\right)^3 - 1 \quad (4)$$

$$\delta_{\text{lin}}(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\text{coll}}}\right)^{2/3} \quad (5)$$

connects full nonlinear result with linear counterpart
$\rightarrow$ maps between the two

- at turnaround
  $\delta_{\text{nonlin}} = (6\pi)^2/4^3 - 1 = 4.6$
  but $\delta_{\text{lin}} = 1.06$
- including virialization (PS6):
  $\delta_{\text{nonlin}} \approx 180$, but $\delta_{\text{lin}} = 1.69$
  defines a critical linear overdensity

Q: why useful?

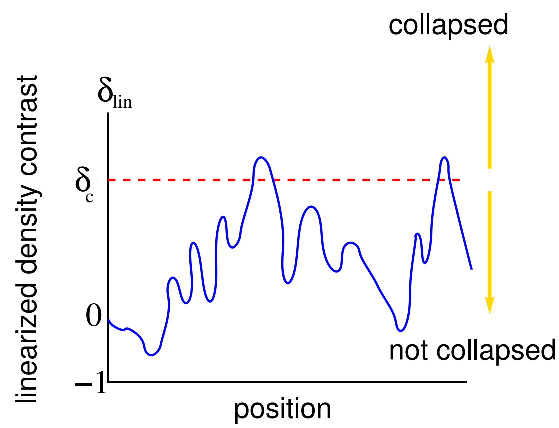


Strategy: *given initial linear density field* δ_i

- evolve perturbations with linear growth $\delta_{\rm lin}(t)$
- identify linearly extrapolated perturbations with $\delta_{\rm lin}(t) > 1.69$

  $\Rightarrow$ *these will be collapsed objects by time* t*!*

lesson: in *linearized* $\delta_{\rm lin}(t_0)$
a "cut" at $\delta_c = 1.69$
divides virialized vs nonvirialized



also: in a *nonlinear field*, can use $\delta_{\rm vir} \sim 180$
as working *definition* of collapsed structure
good for comparing theory, observation *Q: procedure?*

Nonlinear Evolution: Lessons from Spherical Collapse

Qualitatively

▷ overdensity evolves as closed "subuniverse"

▷ starts expanding, but slower than cosmic background
pulls away from Hubble flow: reach max expansion, then turnaround

▷ virialize → form bound object

▷ no further expansion, except due to accretion, merging

Quantitatively

▷ can compute both $\delta_{\rm lin}(t)$ and exact $\delta(t)$
gives mapping from *easy* to (more) *correct*

▷ collapse/virialization when $\boxed{\delta_{\rm lin} = 1.69 \text{ and } \delta = 18\pi^2 \simeq 180}$
recipe for forecasting structures in initial field $\delta_{\rm init} \ll 1$
recipe for defining halos: region surrounding density peak
and having overdensity $\delta\rho/\rho \sim 180$

★ Given these, can devise analytical tools to describe distribution of structures

Intermission: Movies!

Theory of Cosmological Perturbations

Treat structure formation as initial value problem

- given *initial conditions*: "seeds"
  i.e., adopt spectrum of primordial density perturbations
  prescription for initial $\rho_i(\vec{x})$, $i \in$ baryons, radiation, DM, DE...
  e.g., inflation: scale invariant, gaussian, adiabatic
- follow *time evolution* of $\rho_i(\vec{x})$–i.e., δ_i for each species i
- compare with observed measures of structure

★ agreement (or lack thereof) constrains primordial seeds
  e.g., dark matter, inflation, quantum gravity, ...

We want to describe dynamics of cosmic inhomogeneities

Q: which forces relevant? which irrelevant? which scary?

Dynamics Cosmological Perturbations: Overview

Forces/interactions in perturbed, inhomogeneous universe
involve same cosmic particle/field content
as smooth/unperturbed universe
but: can manifest in new/different ways due to spatial variations

Definitely relevant forces on perturbations

- *gravity*: overdensities have extra attraction
  over that of "background" FRW universe
- *pressure*: baryons have thermal pressure $P = nkT$
  photons exert radiation pressure on baryons pre-decoupling
  pressure *gradients* present, unlike in homog. background

Probably irrelevant forces on perturbations (will ignore)

- neutrino interactions with self, other species
- dark matter non-gravity interactions with self, or other species

Scary forces on perturbations (will ignore for now, but worry about)

- if dark energy is a field ϕ, perturbations $\delta\phi$ exert inhomogeneous *negative* pressure
  why scary? unknown underlying physics
- magnetic fields $\rightarrow$ pressure, MHD forces
  why scary? unknown initial conditions (primordial B?)

At minimum: we will want to describe baryons & dark matter as inflationary perturbations grow thru radiation, matter eras
$\rightarrow$ *gravity* and photon, baryon *pressure* mandatory
schematically:

$$\text{acceleration} = -\text{gravity} + \text{pressure} \tag{6}$$

Q: implications for baryons vs dark matter?

For the species and forces we choose to follow:
Q: how can these be described exactly? approximately?
Q: what formalism needed?

Dynamics of Cosmological Perturbations: Toolbox

need dynamics of inhomogeneous **“fluids”**
in expanding FLRW background

⋆ full treatment: general relativistic perturbation theory
mandatory for some results *Q: which?*

⋆ good-enough treatment: Newtonian dynamics is FLRW
as usual, benefits: intuition & simplicity
costs: limited range of validity

Newtonian Fluid Dynamics & Self-Gravity

Each cosmic species is "fluid" described by fields

- mass density $\rho(\vec{x}, t)$
- velocity $\vec{v}(\vec{x}, t)$
- pressure $P(\vec{x}, t)$: from equation of state $P = P(\rho, T)$

In Newtonian limit: dynamics governed by **fluid equations**

1. mass conservation: continuity
2. "$F = ma$": Euler
3. inverse square gravity: Poisson

Fluid Equations: Mass Conservation

1. *mass conservation* (continuity)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad (7)$$

- formally identical to EM continuity equation *Q: why?*
- coordinates: fixed in space (don't move with fluid: Eulerian)
  Q: if fluid at rest, $\vec{v} = 0$, what happens?
- in coordinates that move with fluid:
  need "*convective derivative*"

$$\begin{aligned} d\rho(\vec{x}, t)/dt &= (\partial_t + \dot{x}_i \partial_i)\rho & (8) \\ &= \partial_t \rho + \vec{v} \cdot \nabla \rho & (9) \\ &\overset{\text{cont}}{=} -\rho \nabla \cdot \vec{v} & (10) \end{aligned}$$

Q: when does ρ increase? why?

Fluid Equations: Forces

include forces:

- pressure P
- gravity: acceleration $\vec{g} = -\nabla\Phi$, potential Φ

2. Euler Equation: "$F = ma$"

$$\begin{aligned} \rho d\vec{v}/dt &= \rho\partial_t\vec{v} + \rho\vec{v}\cdot\nabla\vec{v} && (11) \\ &= -\nabla P + \rho\vec{g} = -\nabla P - \rho\nabla\Phi && (12) \end{aligned}$$

Q: what if $\vec{g} = \Phi = P = 0$?

Q: what if $\vec{g} = \Phi = 0$, and spatially uniform $P(\vec{x}) = P_0$?

Q: what if $P = 0$ but $\vec{g} \neq 0$? Hint–this is dark matter's life!

Q: what direction is pressure force?

Q: what determines Φ?

Fluid Equations: Newtonian Gravity

3. Newtonian gravity: inverse square law
encoded in Poisson equation

$$\nabla^2 \Phi = 4\pi G\rho \tag{13}$$

equivalent to Gauss' law $\nabla \cdot \vec{g} = -4\pi G\rho$

To summarize: fluid equations

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 & (14)\\
\rho \partial_t \vec{v} + \rho \vec{v} \nabla \cdot \vec{v} &= -\nabla P - \rho \nabla \Phi & (15)\\
\nabla^2 \Phi &= 4\pi G\rho & (16)
\end{aligned}$$

These are general (albeit Newtonian only)
→ now apply to the Universe

Linear Theory 0: Newtonian, Non-expanding

consider *static*, uniform (infinite) distribution of matter
and introduce small perturbations

$$\rho(\vec{x}) = \rho_0 \ [1 + \delta(\vec{x})] \quad (17)$$
$$v(\vec{x}) = \vec{u}(\vec{x}) \quad (18)$$
$$\Phi_{\rm grav}(\vec{x}) = \Phi_0 + \Phi_1(\vec{x}) \quad (19)$$

where $\delta \ll 1$, and $\Phi_1, \vec{u}$ "small"

we want: time development of (initially) small perturbations
following Sir James Jeans
many key ideas of full expanding-Universe GR result
already appear here!

Newtonian fluid equations: continuity (mass conservation)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \quad (20)$$
$$\rho_0 \dot{\delta} + \rho_0 \nabla \cdot \vec{u} \approx 0 \quad (21)$$

Euler ("$F = ma$");

$$\rho d\vec{v}/dt = \rho\partial_t\vec{v} + \rho\vec{v}\cdot\nabla\vec{v} \;=\; -\nabla p - \rho\nabla\Phi \tag{22}$$

$$\rho_0\ddot{\vec{u}} \;\approx\; -\rho_0 c_s^2\nabla\delta - \rho_0\nabla\Phi_1 \tag{23}$$

where **adiabatic sound speed** $c_s^2 = \partial p/\partial\rho$

Gravity: Poisson (Gauss' law = inverse square force)

$$\nabla^2\Phi \;=\; 4\pi G\rho \tag{24}$$

$$\nabla^2\Phi_1 \;\approx\; 4\pi G\rho_0\delta \tag{25}$$

note inconsistency=cheat! $\nabla^2\Phi_0 \neq 4\pi G\rho_0$: "*Jeans swindle*"

can combine to single eq for linearized density contrast:

$$\partial_t^2\delta - c_s^2\nabla^2\delta = 4\pi G\rho_0\delta \tag{26}$$

Q: behavior for pressureless fluid? "switched-off" gravity? physical significance? important scales?

Density contrast evolves as

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{27}$$

solutions are of the form

$$\delta(t, \vec{x}) = A e^{i(\omega t - \vec{k}\cdot\vec{x})} \equiv D(t)\ \delta_0(\vec{x}) \tag{28}$$

where $\delta_0(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}$ is init Fourier amp
and time evolution is set by exponent $\omega(k)$:

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) = \left(\frac{c_s}{k_J}\right)^2 \left[\left(\frac{\lambda_J}{\lambda}\right)^2 - 1\right] \tag{29}$$

key scale: **Jeans length**

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s} \qquad \lambda_J = \frac{c_s}{\sqrt{G\rho_0/\pi}} \sim c_s \tau_{\rm freefall} \tag{30}$$

associate Jeans mass: $M(\lambda_J/2) = 4\pi/3\ \rho_0 (\pi/k_J)^3 \sim c_s^3 / G^{3/2} \rho_0^{1/2}$—
Q: physically, what expect for $\lambda < \lambda_J$? $\lambda > \lambda_J$?

perturbation growth $\delta_k(t) = \delta_k(t_0) e^{i\omega t}$, with

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) \tag{31}$$

Jeans length $\sim c_s \tau_{\rm freefall}$: sound travel distance in freefall time
$\rightarrow \lambda/\lambda_J \sim$ number of pressure wave crossings during freefall

if $k > k_J$ so $\lambda < \lambda_J$, small scales: pressure can repel contraction
ω real: oscillations about hydrostatic equilib

if $k < k_J$ so $\lambda > \lambda_J$, large scales: pressure ineffective
ω imaginary, exponential collapse
runaway perturbation growth $D(t) = e^{\omega t} \sim e^{+t/t_{\rm freefall}}$
(also an uninteresting decaying mode $e^{-\omega t}$)

Q: but what about expanding Universe?
should grav instability be stronger or weaker?

Linear Theory I: Newtonian Analysis in Expanding U.

Recall: Newtonian analysis legal for small scales, weak gravity
$\rightarrow$ okay for linear analysis inside Hubble length
apply to **matter-dominated U.**

Coordinate choices

Eulerian time-indep grid $\vec{x}$ fixed in physical space
expansion moves unperturbed fluid elts past as $\vec{x}(t) = a(t)\vec{r}$

Lagrangian coords $\vec{r}$ time-indep but expand in physical space
following fluid element: *locally* comoving

$\Rightarrow$ spatial gradients: $\nabla_{\vec{x}} = (1/a)\nabla_{\vec{r}}$

Unperturbed (zeroth order) eqs,
using $\rho_0 = \rho_0(t)$, $\vec{v}_0 = \frac{\dot{a}}{a}\vec{x} = \dot{a}\vec{r}$

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \vec{v}) \;=\; \dot{\rho_0} + \rho_0 \frac{\dot{a}}{a} \nabla_{\vec{x}} \cdot \vec{x} = 0 \quad (32)$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0 \qquad \Rightarrow \rho_0 \propto a^{-3} \quad (33)$$

Poisson:

$$\nabla^2\Phi_0 = \frac{1}{x^2}\partial_x(x\partial_x\Phi_0) = 4\pi G\rho_0 \;\Rightarrow\; \Phi_0 = \frac{2\pi G\rho_0}{3}x^2 = \frac{2\pi G\rho_0}{3}a^2r^2$$

$$\nabla_{\vec{x}}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{x} \qquad \nabla_{\vec{r}}\Phi_0 = \frac{4\pi G\rho_0}{3}a\vec{r}$$

Euler

$$d(\dot{a}\vec{r})/dt = \ddot{a}\vec{r} = \frac{\ddot{a}}{a}\vec{x} = -\frac{4\pi G\rho_0}{3}\vec{x} \quad (34)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{3} \quad (35)$$

Fried accel; with continuity $\rightarrow$ Friedmann

Zeroth order fluid equations $\rightarrow$ usual expanding U
in non-rel approximation

Now add perturbations $\rho_1 = \rho_0 \delta$, $\vec{v}_1$, Φ_1

simplest to use comoving (Lagrangian) coords
follow observers in unperturbed Hubble flow
perturbation fluid elements $\vec{x}(t) = a(t)\vec{r}(t)$
peculiar fluid velocity $\vec{v}_1(t) = a(t)\vec{u}(t)$

plug in, keep only terms linear in perturbations ($\nabla = \nabla_{\vec{r}}$)
$\rightarrow$ *perturbation evolution to first (leading, linear) order*

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla \delta p}{\rho_0} \qquad (36)$$

$$\dot{\delta} = -\nabla \cdot \vec{u} \qquad (37)$$

consider the case of $\Phi_1 = 0$ and $\delta p = 0$, but initial $\vec{u} \neq 0$
Q: what does this represent physically? what happens? why?
Q: implications for the situation when $\Phi_1 \neq 0$ and $\delta\rho \neq 0$?

Velocity Perturbation Evolution

peculiar velocity $\vec{v_1} = a(t)\ \vec{u}$ evolves as

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \tag{38}$$

if no pressure nor density perturbations
then $\dot{u} = -2Hu$, and so $u \propto 1/a^2$
and physical speed evolves as $v_1 \propto 1/a$

but recall: long ago derived FLRW test particle speed
evolves as $\vec{v}(t) = \vec{v}_0/a(t)$
→ pressureless fluid's peculiar vel redshifts same as free particle
→ expansion acts as "drag" on particles

if $\Phi_1, \delta p \neq 0$: Hubble "drag" still present
removes kinetic energy from collapsing objects
allows total energy to change (decrease) with time
→ *binding increases!*

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes
e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\boxed{\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k} \tag{39}$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "friction" or "drag" $-2H\dot{\delta}$ opposes density growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger