Astro 507

Lecture 36

April 24, 2020

Announcements:

- **Problem Set 6 extended Monday April 27**
  after this: final Problem Set due Finals Week
  recall: lowest PF and PS dropped
  can drop Finals Week PS

- **Preflight 6b due next Friday May 1**
  draft your Wikipedia upgrade, post for comments
  have fun, ask if you need advice/help

Last time & PS6 Q2: comparing two spherical collapse solutions

$$\delta_{\rm nonlin}(t) = \left(\frac{a_{\rm bg}}{a_{\rm nonlin}}\right)^3 - 1 \quad (1)$$

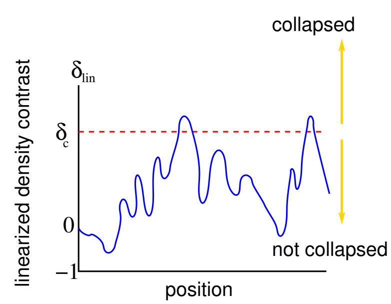
$$\delta_{\rm lin}(t) \approx \frac{3}{20}\left(\frac{12\pi t}{t_{\rm coll}}\right)^{2/3} \quad (2)$$

at any time t this maps between
full nonlinear result with linearized approximation

Strategy: *given initial linear density field* δ_i

- evolve perturbations with linear growth $\delta_{\rm lin}(t)$
- identify linearly extrapolated perturbations with $\delta_{\rm lin}(t) > 1.69$
  $\Rightarrow$ *these will be collapsed objects by time* t*!*

lesson: in *linearized* $\delta_{\rm lin}(t_0)$
a "cut" at $\delta_c = 1.69$
divides virialized vs nonvirialized



Theory of Cosmological Perturbations

Treat structure formation as initial value problem

- given *initial conditions*: "seeds"
  i.e., adopt spectrum of primordial density perturbations
  prescription for initial $\rho_i(\vec{x})$, $i \in$ baryons, radiation, DM, DE...
  e.g., inflation: scale invariant, gaussian, adiabatic
- follow *time evolution* of $\rho_i(\vec{x})$–i.e., δ_i for each species i
- compare with observed measures of structure

$\star$ agreement (or lack thereof) constrains primordial seeds
e.g., dark matter, inflation, quantum gravity, ...

We want to describe dynamics of cosmic inhomogeneities

Q: which forces relevant? which irrelevant? which scary?

Dynamics Cosmological Perturbations: Overview

Forces/interactions in perturbed, inhomogeneous universe
involve same cosmic particle/field content
as smooth/unperturbed universe
but: can manifest in new/different ways due to spatial variations

Definitely relevant forces on perturbations

- *gravity*: overdensities have extra attraction
  over that of "background" FRW universe
- *pressure*: baryons have thermal pressure $P = nkT$
  photons exert radiation pressure on baryons pre-decoupling
  pressure *gradients* present, unlike in homog. background

Probably irrelevant forces on perturbations (will ignore)

- neutrino interactions with self, other species
- dark matter non-gravity interactions with self, or other species

Scary forces on perturbations (will ignore for now, but worry about)

- if dark energy is a field ϕ, perturbations $\delta\phi$ exert inhomogeneous *negative* pressure
  why scary? unknown underlying physics
- magnetic fields $\rightarrow$ pressure, MHD forces
  why scary? unknown initial conditions (primordial B?)

At minimum: we will want to describe baryons & dark matter as inflationary perturbations grow thru radiation, matter eras
$\rightarrow$ *gravity* and photon, baryon *pressure* mandatory
schematically:

$$\text{acceleration} = -\text{gravity} + \text{pressure} \tag{3}$$

Q: implications for baryons vs dark matter?

For the species and forces we choose to follow:
Q: how can these be described exactly? approximately?
Q: what formalism needed?

Dynamics of Cosmological Perturbations: Toolbox

need dynamics of inhomogeneous **"fluids"**
in expanding FLRW background
⋆ full treatment: general relativistic perturbation theory
mandatory for some results *Q: which?*
⋆ good-enough treatment: Newtonian dynamics is FLRW
as usual, benefits: intuition & simplicity
costs: limited range of validity

Newtonian Fluid Dynamics & Self-Gravity

Each cosmic species is "fluid" described by fields

- mass density $\rho(\vec{x},t)$
- velocity $\vec{v}(\vec{x},t)$
- pressure $P(\vec{x},t)$: from equation of state $P = P(\rho, T)$

In Newtonian limit: dynamics governed by **fluid equations**

1. mass conservation: continuity
2. "$F = ma$": Euler
3. inverse square gravity: Poisson

Fluid Equations: Mass Conservation

1. *mass conservation* (continuity)

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \qquad (4)$$

- formally identical to EM continuity equation *Q: why?*
- coordinates: fixed in space (don't move with fluid: Eulerian)
  Q: if fluid at rest, $\vec{v} = 0$, *what happens?*
- in coordinates that move with "fluid element":
  need "*convective derivative*"

$$\begin{aligned} d\rho(\vec{x}, t)/dt &= (\partial_t + \dot{x}_i \partial_i)\rho & (5) \\ &= \partial_t \rho + \vec{v} \cdot \nabla \rho & (6) \\ &\overset{\text{cont}}{=} -\rho \nabla \cdot \vec{v} & (7) \end{aligned}$$

∞

Q: when does ρ *increase? why?*

Fluid Equations: Forces

include forces:

- pressure P
- gravity: acceleration $\vec{g} = -\nabla\Phi$, potential Φ

2. Euler Equation: "$F = ma$" in a volume element

$$\rho d\vec{v}/dt = -\nabla P + \rho\vec{g} \tag{8}$$

$$\rho\partial_t\vec{v} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P - \rho\nabla\Phi \tag{9}$$

Q: what if $\vec{g} = \Phi = P = 0$*?*

Q: what if $\vec{g} = \Phi = 0$*, and spatially uniform* $P(\vec{x}) = P_0$*?*

Q: what if $P = 0$ *but* $\vec{g} \neq 0$*? Hint–this is dark matter's life!*

Q: what direction is pressure force?

Q: what determines Φ*?*

Fluid Equations: Newtonian Gravity

3. Newtonian gravity: inverse square law
encoded in Poisson equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{10}$$

equivalent to Gauss' law $\nabla \cdot \vec{g} = -4\pi G \rho$

To summarize: fluid equations

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 & (11) \\ \rho \partial_t \vec{v} + \rho \vec{v} \nabla \cdot \vec{v} &= -\nabla P - \rho \nabla \Phi & (12) \\ \nabla^2 \Phi &= 4\pi G \rho & (13) \end{aligned}$$

These are general (albeit Newtonian only)
$\rightarrow$ now apply to the Universe

Linear Theory 0: Newtonian, Non-expanding

consider *static*, uniform (infinite) distribution of matter
and introduce **perturbations**

$$\begin{aligned} \rho(\vec{x}) &= \rho_0 \; [1 + \delta(\vec{x})] & (14) \\ v(\vec{x}) &= \vec{u}(\vec{x}) & (15) \\ \Phi_{\rm grav}(\vec{x}) &= \Phi_0 + \Phi_1(\vec{x}) & (16) \end{aligned}$$

focus on linear regime–**small perturbations**: $\delta \ll 1$, and $\Phi_1, \vec{u}$

we want: time development of (initially) small perturbations
following Sir James Jeans
many key ideas of full expanding-Universe GR result
already appear here!

Fluid equations: continuity (mass conservation), to first order

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 & (17) \\ \rho_0 \dot{\delta} + \rho_0 \nabla \cdot [(1+\delta)\vec{u}] &\approx \rho_0 \dot{\delta} + \rho_0 \nabla \cdot \vec{u} = 0 & (18) \end{aligned}$$

Euler ("$F = ma$");

$$\rho d\vec{v}/dt = \rho\partial_t\vec{v} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla p - \rho\nabla\Phi \tag{19}$$

$$\rho_0\ddot{\vec{u}} \approx -\rho_0 c_s^2\nabla\delta - \rho_0\nabla\Phi_1 \tag{20}$$

where **adiabatic sound speed** $c_s^2 = \partial p/\partial\rho$

Gravity: Poisson (Gauss' law = inverse square force)

$$\nabla^2\Phi = 4\pi G\rho \tag{21}$$

$$\nabla^2\Phi_1 \approx 4\pi G\rho_0\delta \tag{22}$$

note inconsistency=cheat! $\nabla^2\Phi_0 \neq 4\pi G\rho_0$: "*Jeans swindle*"

can combine to single eq for linearized density contrast:

$$\partial_t^2\delta - c_s^2\nabla^2\delta = 4\pi G\rho_0\delta \tag{23}$$

Q: behavior for pressureless fluid? "switched-off" gravity?
physical significance? important scales?

Density contrast evolves as

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{24}$$

note wave operator! seek wavelike solutions

$$\delta(t, \vec{x}) = A e^{i(\omega t - \vec{k}\cdot\vec{x})} \equiv D(t)\ \delta_0(\vec{x}) \tag{25}$$

with Fourier amplitude $\delta_0(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}$ for wavevector $|\vec{k}| = 2\pi/\lambda$ and time evolution is set by exponent $\omega(k)$:

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) = \left(\frac{c_s}{k_J}\right)^2 \left[\left(\frac{\lambda_J}{\lambda}\right)^2 - 1\right] \tag{26}$$

key scale: **Jeans length**

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s} \quad \lambda_J = \frac{c_s}{\sqrt{G\rho_0/\pi}} \sim c_s \tau_{\rm freefall} \tag{27}$$

associate Jeans mass: $M(\lambda_J/2) = 4\pi/3\ \rho_0 (\pi/k_J)^3 \sim c_s^3/G^{3/2}\rho_0^{1/2}$

Q: physically, what expect for $\lambda < \lambda_J$*?* $\lambda > \lambda_J$*?*

perturbation growth $\delta_k(t) = \delta_k(t_0)e^{i\omega t}$, with

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0 \equiv c_s^2(k^2 - k_J^2) \tag{28}$$

Jeans length $\sim c_s \tau_{\text{freefall}}$: sound travel distance in freefall time
$\rightarrow \lambda/\lambda_J \sim$ number of pressure wave crossings during freefall

if $k > k_J$ so $\lambda < \lambda_J$, **small scales**: pressure can repel contraction
ω real: **oscillations** about hydrostatic equilib

if $k < k_J$ so $\lambda > \lambda_J$, **large scales:** pressure ineffective
ω imaginary, exponential collapse
runaway perturbation growth $\delta(t) \sim e^{\omega t} \sim e^{+t/t_{\text{freefall}}}$
(also an uninteresting decaying mode $e^{-\omega t}$)

14 *Q: but what about expanding Universe?*
should grav instability be stronger or weaker?

Intermission: Questions?

Linear Theory I: Newtonian Analysis in Expanding U.

Recall: Newtonian analysis legal for small scales, weak gravity
$\rightarrow$ okay for linear analysis inside Hubble length
apply to **matter-dominated U.**

Coordinate choices
Eulerian time-indep grid $\vec{x}$ fixed in physical space
expansion moves unperturbed fluid elts past as $\vec{x}(t) = a(t)\vec{r}$
Lagrangian coords $\vec{r}$ time-indep but expand in physical space
following fluid element: *locally* comoving
$\Rightarrow$ spatial gradients: $\nabla_{\vec{x}} = (1/a)\nabla_{\vec{r}}$

Unperturbed (zeroth order) eqs,
using $\rho_0 = \rho_0(t)$, $\vec{v}_0 = \frac{\dot{a}}{a}\vec{x} = \dot{a}\vec{r}$

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \vec{v}) = \dot{\rho_0} + \rho_0 \frac{\dot{a}}{a} \nabla_{\vec{x}} \cdot \vec{x} = 0 \tag{29}$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0 \qquad \Rightarrow \rho_0 \propto a^{-3} \tag{30}$$

Poisson:

$$\nabla^2\Phi_0 = \frac{1}{x^2}\partial_x(x\partial_x\Phi_0) = 4\pi G\rho_0 \;\Rightarrow\; \Phi_0 = \frac{2\pi G\rho_0}{3}x^2 = \frac{2\pi G\rho_0}{3}a^2r^2$$

$$\nabla_{\vec{x}}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{x} \qquad \nabla_{\vec{r}}\Phi_0 = \frac{4\pi G\rho_0}{3}a\vec{r}$$

Euler

$$d(\dot{a}\vec{r})/dt = \ddot{a}\vec{r} = \frac{\ddot{a}}{a}\vec{x} = -\frac{4\pi G\rho_0}{3}\vec{x} \tag{31}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{3} \tag{32}$$

Fried accel; with continuity $\rightarrow$ Friedmann

Zeroth order fluid equations $\rightarrow$ usual expanding U
in non-rel approximation

Now add perturbations $\rho_1 = \rho_0 \delta$, $\vec{v}_1$, Φ_1

simplest to use comoving (Lagrangian) coords
follow observers in unperturbed Hubble flow
perturbation fluid elements $\vec{x}(t) = a(t)\vec{r}(t)$
peculiar fluid velocity $\vec{v}_1(t) = a(t)\vec{u}(t)$

plug in, keep only terms linear in perturbations $(\nabla = \nabla_{\vec{r}})$
$\rightarrow$ *perturbation evolution to first (leading, linear) order*

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla \delta p}{\rho_0} \quad (33)$$

$$\dot{\delta} = -\nabla \cdot \vec{u} \quad (34)$$

consider the case of $\Phi_1 = 0$ and $\delta p = 0$, but initial $\vec{u} \neq 0$
Q: what does this represent physically? what happens? why?
Q: implications for the situation when $\Phi_1 \neq 0$ and $\delta\rho \neq 0$?

Velocity Perturbation Evolution

peculiar velocity $\vec{v_1} = a(t)\ \vec{u}$ evolves as

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \tag{35}$$

if no pressure nor density perturbations
then $\dot{u} = -2Hu$, and so $u \propto 1/a^2$
and physical speed evolves as $v_1 \propto 1/a$

but recall: long ago derived FLRW test particle speed
evolves as $\vec{v}(t) = \vec{v}_0/a(t)$
$\rightarrow$ pressureless fluid's peculiar vel redshifts same as free particle
$\rightarrow$ expansion acts as "drag" on particles

if $\Phi_1, \delta p \neq 0$: Hubble "drag" still present
removes kinetic energy from collapsing objects
allows total energy to change (decrease) with time
$\rightarrow$ *binding increases!*

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes
e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k \quad (36)$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "friction" or "drag" $-2H\dot{\delta}$ opposes density growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger

Unstable Modes: Matter-Dominated U

Consider **large scales** $\lambda \gg \lambda_J$

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k \approx 4\pi G\rho_0\delta_k \tag{37}$$

in *Matter-dominated U:* $8\pi G\rho/3 = H^2 = (2/3t)^{-2} = 4/9t^2$, so

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{38}$$

Q: how many independent solutions? how to solve?

Matter-dominated U, large scales:

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{39}$$

eq homogeneous in $t \to$ try *power law solution*

trial $\delta \sim t^s$ works if

$$s(s-1) + 4s/3 - 2/3 = 0 \tag{40}$$

solutions $s = 2/3, -1$:

growing and decaying modes

$$\delta_+(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} \quad ; \quad \delta_-(t) = \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1} \tag{41}$$

- growing mode dominates
- Hubble friction: exponential collapse softened to power law

★ Note: solutions indep of k *Q: why a big deal?*

Linear Growth Factor

each unstable Fourier mode grows with time as

$$\delta_k(t) \propto D(t) \sim t^{2/3} \sim a \sim \eta^2_{\text{conform}} \tag{42}$$

growth independent of wavenumber k

- in k-space, all unstable modes grow by same factor and transform to real space, find
- on large scales (but still subhorizon)

$$\delta(t, \vec{x}_{\text{large}}) \simeq D(t)\delta(t_i, \vec{x}_{\text{large}}) \tag{43}$$

$\Rightarrow$ entire density contrast pattern grows with same amplification:

$\Rightarrow$ **linear grow factor** $D(t)$ applies to whole field

Q: what would this look like for $\delta(x)$?

Applications to CMB: Naïve Inferences

before decoupling: pressure dominated by photons
$\rightarrow$ expect oscillations – and see them!
after decoupling: growing mode

CMB anisotropies are a snapshot
of perturbations at last scattering
can quantify level: $(\delta T/T)_{\mathrm{ls}} \sim 10^{-5}$ at $z_{\mathrm{ls}} \sim 1100$

But matter has $\rho \propto a^{-3} \propto T^3$, so $\delta\rho/\rho = 3\delta T/T$
$\rightarrow$ $\delta_{\mathrm{obs}}(z = 1100) \sim 3 \times 10^{-5}$ at last scattering
So today, expect fluctuations of size

$$\delta_0 = \frac{D_0}{D_{\mathrm{ls}}}\delta_{\mathrm{ls}} = \frac{a_0}{a_{\mathrm{ls}}}\delta_{\mathrm{ls}} = (1 + z_{\mathrm{ls}})\delta_{\mathrm{ls}} \sim 0.05 \ll 1 \tag{44}$$

Should still be very small–no nonlinear structures, such as us!
Q: obviously wrong–egregiously naïve! What's the flaw?
What's the fix?

Perturbation Growth: Dark Matter vs Baryons

dark matter: pressureless
$\rightarrow$ all k modes unstable if inside Hubble length
but: perturbations grow verry sloooowly during radiation era
$\rightarrow$ DM structures begin formation at matter-radiation equality
then $\delta_{\rm m}(t) = \delta_{\rm m,init}\ D(t)$ with $D(t) \propto a(t) \propto t^{2/3}$

baryons: until recomb, tightly coupled to photons
$\rightarrow$ feel huge photon pressure $P_\gamma \propto T^4$
$\rightarrow$ sound speed $c_s \sim c/\sqrt{3}$ huge!
so all sub-horizon modes stable! just oscillate
$\rightarrow$ relativistic pressure-mediated (i.e., acoustic) standing waves!
oscillation frequency $\nu = c_s/\lambda$:
small-scale modes oscillate many times
largest-scale modes $\lambda = c_s \eta_{\rm hor}$ oscillates only once

Pre-Recombination: Acoustic Oscillations

Baryons in DM-dominated background

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b \simeq 4\pi G\rho\delta_{dm} - \frac{k^2 c_s^2}{a^2}\delta_b \sim \frac{\delta_{dm}}{t^2} - \frac{k^2 c_s^2}{a^2}\delta_b \tag{45}$$

key comparison: mode scale $\lambda \sim k^{-1}$
vs **comoving sound horizon** $c_s t/a = d_{s,com}$

for *large scales* $k c_s t/a \ll 1$: *baryons follow DM*
for *small scales* $k c_s t/a \gg 1$: *baryons oscillate*, as

$$\delta_b \sim \frac{1}{\sqrt{a c_s k}} e^{i\int k c_s d\eta} \tag{46}$$

(PS 6) where $d\eta = dt/a$ is conformal time

Q: for fixed k, what is δ time behavior?
Q: at fixed t, what is δ pattern vs k?
Q: what sets largest λ that oscillates?

baryonic perturbations do not grow, but oscillate:

$$\delta_b \sim \frac{1}{\sqrt{ac_s k}} e^{i \int k c_s d\eta} \tag{47}$$

to simplify, imagine constant c_s, $\delta_b \sim e^{ikc_s\eta}$

at fixed k, sinusoidal oscillations

phase counts number of cycles $N = kc_s\eta/2\pi = c_s\eta/\lambda$

oscillation frequency: $\omega \sim kc_s \sim c_s/\lambda \propto 1/\lambda$

at fixed $t \rightarrow$ fixed η:

small λ and large $k \rightarrow$ rapid oscillations

largest oscillations at scale $\lambda \sim c_s\eta \sim c_s t/a$: *sound horizon*

Q: when do oscillations stop? observable signature?

Director's Cut Extras

Non-relativistic Cosmic Kinematics

gas particles have random thermal speeds, momenta
how are these affected by cosmic expansion?

Classical picture:
consider non-rel free* particle moving w.r.t. comoving frame
$\vec{\ell}_{\rm com}(t) \neq const$, and so $\vec{\ell}_{\rm phys} = a(t)\ell_{\rm com}(t)$:

$$\begin{aligned} \vec{v} = d\vec{\ell}_{\rm phys}/dt &= \dot{a}(t)\ell_{\rm com}(t) &+& \quad a(t)\dot{\ell}_{\rm com}(t) \\ &= H\vec{\ell}_{\rm phys} &+& \quad \vec{v}_{\rm pec} \\ &= \text{Hubble flow} &+& \quad \text{peculiar velocity} \end{aligned}$$

Note that peculiar velocity v is always w.r.t. the comoving frame–i.e., the particle speed compared to that of a stationary fundamental observer *at the same point*

*i.e., except for gravitation

consider a comoving observer at the origin, $\vec{x} = 0$
in time δt, a particle moves w.r.t. comov frame
physical dist $\delta\vec{x}_{\text{phys}} = \vec{v}_{\text{pec}}\delta t$

but due to Hubble flow, a comoving (fundamental) observer at $\delta\vec{x}_{\text{phys}}$ is moving away from the origin at speed $\vec{v}_{\text{com}} = H\delta\vec{x}_{\text{phys}}$

thus the new speed of the particle relative to its new comoving neighbor is given by the relative velocity
$\vec{v}'_{\text{pec}} = \vec{v}_{\text{pec}} - \vec{v}_{\text{com}}$
(where we used the non-rel velocity addition law)
and so the peculiar velocity *changes* by

$$\delta\vec{v}_{\text{pec}} = -H\delta\vec{x}_{\text{phys}} = -\frac{\dot{a}}{a}\vec{v}_{\text{pec}}\delta t = -\frac{\delta a}{a}\vec{v}_{\text{pec}} \tag{48}$$

Q: physical implications?

$\delta v_{\text{pec}}/v_{\text{pec}} = -\delta a/a \Rightarrow$ physical peculiar velocity $v_{\text{pec}} \propto 1/a$:

- $m v_{\text{non-rel}} = p_{\text{non-rel}} = p_0/a$
- comoving peculiar velocity $d\ell_{\text{com}}/dt \propto 1/a^2$
  slowdown w.r.t. comoving frame: velocity "decays"
  not a "cosmic drag" but rather kinematic effect
  due to struggle to overtake receding of cosmic milestones

Quantum picture:

recall for photons, $p_{\text{rel}} = h/\lambda \sim 1/a$ (de Broglie)
but de Broglie holds for matter too: $p_{\text{non-rel}} = h/\lambda_{deB} \sim 1/a$
$\Rightarrow$ again, $p_{\text{non-rel}} = p_0/a$

true in general, now apply to thermal gas non-relativistic gas: Maxwell-Boltzmann

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} a^{-3} \int d^3p_0 \; e^{-p_0^2/2mka^2T}$$

if occupation number constant (particle conservation)
need $a^2 T(a) = T_0 = const$ and thus $T_{\text{non-rel}} \propto 1/a^2$:

$$T_{\text{non-rel,decoupled}} = \left(\frac{a_{\text{dec}}}{a}\right)^2 T_{\text{decoupling}} = \left(\frac{1+z}{1+z_{\text{dec}}}\right)^2 T_{\text{decoupling}}$$

evaluate for $z_{\text{dec}} = z_{\text{ri}}$: estimate

$$T_{\text{gas,today}} \sim \frac{T_{\gamma,0}}{1+z_{\text{dec,gas}}} \sim 6 \times 10^{-3}\ \text{K} \tag{49}$$

Q: do the experiment...?

Q: what went wrong?

Inhomogeneities: The Spice of Life

So far: we have assumed perfect homogeneity!
If universe strictly homogeneous
indeed would cool to $T_{\rm gas} \ll T_0$

But happily, U. definitely inhomogeneous on small scales!
gravity amplifies density contrast *Q: why?*
"the rich get richer, the poor get poorer"

this allows for motion, condensation of matter
halo formation, mergers, shocks, star formation, quasars, ...
these overdense structures release energy
lead to diversity of cosmic matter and radiation today!

But how did we get the inhomogeneities?
And what set the primordial composition of baryons?
$\rightarrow$ events in the very early Universe...

Momentum Redshifting: Rigorously

the preceding heuristic arguments give the right result, but to obtain this rigorously requires General Relativity (if you haven't had GR yet, never mind)

in GR: a free particle's motion is a geodesic
so 4-momentum $p^{\mu} = m dx^{\mu}/ds = m(\gamma, \gamma\vec{v}) = (E, \vec{p})$ changes as

$$p^{\alpha}\nabla_{\alpha}p^{\mu} = p^{\alpha}\partial_{\alpha}p^{\mu} + \Gamma^{\mu}_{\alpha\beta}p^{\alpha}p^{\beta} = 0 \qquad (50)$$

and we see that the change in u is due to the connection term Γ, i.e., to curvature
$\rightarrow$ curvature tells matter how to move

note: homogeneity hugely simplifies: $p^{\mu} = p^{\mu}(t)$
so $\partial_{\mu}p = 0$ except for $\partial_{t}p = \dot{p}$

consider the $\mu = i \in (x, y, z)$ component of the geodesic eq

$$p^\alpha \partial_\alpha p^i + \Gamma^i_{\alpha\beta} p^\alpha p^\beta = E\dot{p} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta \quad (51)$$

$$= 0 \quad (52)$$

note that in FRW, if we write $ds^2 = dt^2 - h_{ij} dx^i dx^j$
where h_{ij} is the spatial metric, then nonzero $\Gamma^i_{\alpha\beta}$ are

$$\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j \quad (53)$$

where δ^i_j is the Kronecker delta (try it!)

We then have

$$E\dot{p^i} + \frac{\dot{a}}{a} E p^i = 0 \quad (54)$$

and thus

$$d\vec{p}/dt = -\frac{\dot{a}}{a}\vec{p} \quad (55)$$

$$|\vec{p}| \propto \frac{1}{a} \quad (56)$$

Note that this result is completely general, i.e., works for all relativistic p, so

- in non-rel limit, $v \propto 1/a$: vel redshifts, and free particles eventually come to rest wrt the comoving background
- in ultra-rel limit, $v = p/E \approx c$, doesn't redshift, but since $E \approx p$, $E \propto 1/a$: energy redshifts

note classical derivation: didn't need Planck/de Broglie relation $p \propto 1/\lambda$ to show this (though that still works too)

Linear Theory II: Sketch of Relativistic Treatment

see, e.g., Dodelson text, Liddle & Lyth Ch. 14

Recall limits of Newtonian treatment:

- only appropriate for scales $\lambda \ll d_H$: sub-horizon
- relativistic effects like time dilation absent or *ad hoc*

General Relativistic approach to cosmological perturbations

- as in Newtonian analysis, perturb density, velocity
  $\rightarrow$ this perturbs stress-energy
  schematically "$\delta T \approx \delta\rho + \delta P = \delta\rho + c_s^2 \delta\rho$"
- must therefore add small perturbations to metric:
  $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{FRW}} + h_{\mu\nu}$
- these are related by Einstein's Equation
  $G_{\mu\nu} \approx$ "$\partial\partial g^{\mathrm{FRW}} + \partial\partial h$" $= 8\pi G_N T_{\mu\nu} \approx$ "$8\pi G_N(\rho + \delta\rho)$"

Metric Perturbations

Perturbations to metric tensor can be classified as:

- *scalar* – density perturbations couple to these
  these are most important
- *vector* – velocity perturbations couple to these
  these are least important (perturbations decay with time)
- *tensor* – source of gravity waves
  inflationary quantum perturbation excite these modes!

focus on *scalar* perturbations, which modify FRW metric thusly:

$$(ds^2)_{\rm perturbed} = a(\eta)^2 \left[(1 + 2\,\Psi\,) d\eta^2 - (1 - 2\,\Phi\,) \delta_{ij} dx^i dx^j \right] \quad (57)$$

Coordinate freedom $\leftrightarrow$ "gauge" choice $\leftrightarrow$ spacetime "slicing"

 $\Rightarrow$ here: *"conformal Newtonian gauge"*:

- $\Psi(\vec{x}, t), \Phi(\vec{x}, t)$ Schwarzchild-like forms if $a = 1, \dot{a} = 0$

Substitute perturbed metric into Einstein, keep only linear terms in Φ and Ψ, e.g., neglect Φ^2

use conformal time

and go to k-space

- $\nabla_\mu T^{\mu 0} \rightarrow$ "continuity"

$$\frac{d\delta}{d\eta} + ikv + 3\frac{d\Phi}{d\eta} = 0 \qquad (58)$$

- $\nabla_\mu T^{\mu i} \rightarrow$ "Euler"

$$\frac{dv}{d\eta} + \frac{da/d\eta}{a}v + ik\Psi = \text{pressure sources} \qquad (59)$$

- $G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} \rightarrow$ "Poisson"

$$k^2\Phi = -4\pi G a^2 \rho\delta \qquad (60)$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 \text{``}\langle P_x - P_y\rangle\text{''} \qquad (61)$$

expect *anisotropic stress* small: $\langle P_x - P_y\rangle \ll \rho\delta \rightarrow$ $\boxed{\Psi \approx \Phi}$

Recall: conformal time η gives particle horizon

On *sub-horizon* scales $\lambda \sim 1/k \ll \eta$:
relativistic treatment gives back Newtonian result
in fact: justifies our Newtonian treatment

On *super-horizon* scales $\lambda \sim 1/k \gg \eta$:
relativistic treatment still valid
$\rightarrow$ will use this to follow inflationary perturbations
through horizon crossing