Astro 507

Lecture 37

April 27, 2020

Announcements:

- **Problem Set 6 extended to today**
  I will stay on after lecture
- **Preflight 6b due this Friday May 1**
  draft your Wikipedia upgrade, post for comments
  have fun, ask if you need advice/help

Last time: building cosmological perturbation theory
goal: calcul
ate density fluctuation growth in linear regime

Q: for which $\delta = \delta\rho/\rho$ should this approach be valid?
In other words–for what δ should linear theory break down?

Cosmological Perturbation Theory

Perturbative approach:

assumes fluctuations small compared to background

so $|\delta\rho| \ll \rho_0$ and thus $|\delta| \ll 1$

and certainly expect linear theory to fail when $|\delta| \sim 1$

Newtonian fluid equations:

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 && (1)\\
\rho \partial_t \vec{v} + \rho \vec{v}\, \nabla \cdot \vec{v} &= -\nabla P - \rho \nabla \Phi && (2)\\
\nabla^2 \Phi &= 4\pi G \rho && (3)
\end{aligned}
$$

firs step: perturb around

a homogeneous fluid ρ_0 with pressure P_0 at rest $\vec{v}_0 = 0$

Linear Theory 0: Newtonian, Non-expanding

consider *static*, uniform (infinite) distribution of matter
and introduce **perturbations**

$$\begin{aligned} \rho(\vec{x}) &= \rho_0 \, [1 + \delta(\vec{x})] && (4) \\ v(\vec{x}) &= \vec{u}(\vec{x}) && (5) \\ \Phi_{\rm grav}(\vec{x}) &= \Phi_0 + \Phi_1(\vec{x}) && (6) \end{aligned}$$

focus on linear regime–**small perturbations**: $\delta \ll 1$, and $\Phi_1, \vec{u}$

we want: time development of (initially) small perturbations
following Sir James Jeans

many key ideas of full expanding-Universe GR result
already appear here!

Fluid equations: continuity (mass conservation), to first order

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0 && (7) \\ \rho_0 \dot{\delta} + \rho_0 \nabla \cdot [(1+\delta)\vec{u}] &\approx \rho_0 \dot{\delta} + \rho_0 \nabla \cdot \vec{u} = 0 && (8) \end{aligned}$$

Euler ("$F = ma$");

$$\rho d\vec{v}/dt = \rho \partial_t \vec{v} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p - \rho \nabla \Phi \tag{9}$$

$$\rho_0 \dot{\vec{u}} \approx -\rho_0 c_s^2 \nabla \delta - \rho_0 \nabla \Phi_1 \tag{10}$$

where **adiabatic sound speed** $c_s^2 = \partial p / \partial \rho$

Gravity: Poisson (Gauss' law = inverse square force)

$$\nabla^2 \Phi = 4\pi G \rho \tag{11}$$

$$\nabla^2 \Phi_1 \approx 4\pi G \rho_0 \delta \tag{12}$$

note inconsistency=cheat! $\nabla^2 \Phi_0 \neq 4\pi G \rho_0$: "*Jeans swindle*"

PS6: can combine to single eq for linearized density contrast:

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta \tag{13}$$

Q: significance of $\partial_t^2 - \nabla^2$ operator?

Solve for One Wavelength/Mode: Fourier Analysis

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta$$

note wave operator! seek wavelike solutions

$$\delta(t, \vec{x}) = A e^{i(\omega_k t - \vec{k} \cdot \vec{x})} \quad (14)$$

with Fourier amplitude $\delta_0(\vec{x}) = e^{-i\vec{k}\cdot\vec{x}}$ for wavevector $|\vec{k}| = 2\pi/\lambda$
and time evolution is set by exponent ω_k:

$$\omega_k^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) = \left(\frac{c_s}{k_J}\right)^2 \left[\left(\frac{\lambda_J}{\lambda}\right)^2 - 1\right] \quad (15)$$

key length scale: **Jeans length**

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{c_s} \qquad \lambda_J = \frac{c_s}{\sqrt{G\rho_0/\pi}} \sim c_s \tau_{\rm freefall} \quad (16)$$

combine this with background density ρ_0:
characteristic **Jeans mass:** $M(\lambda_J/2) = 4\pi/3 \; \rho_0 (\pi/k_J)^3 \sim c_s^3/G^{3/2}\rho_0^{1/2}$

perturbation evolves as $\boxed{\delta_k(t) = \delta_k(t_{\text{imit}})e^{i\omega_k t}}$:
solution for wavenumber k factorizes

$$\delta_k(t) = \delta_{k,\text{init}}\ D(t) \tag{17}$$

- initial amplitude $\delta_{k,\text{init}}$
- and linear growth factor $D(t) = e^{i\omega_k t}$

whose oscillation frequency is

$$\omega_k^2 = c_s^2 k^2 - 4\pi G \rho_0 \equiv c_s^2 (k^2 - k_J^2) \tag{18}$$

we we define Jeans length $\sim c_s \tau_{\text{freefall}}$:
sound travel distance in freefall time

consider **short-wavelength** modes
with $k > k_J$ so $\lambda < \lambda_J$

Q: how do these mode amplitudes evolve with time?

Short Wavelength Modes

if $k > k_J$ so $\lambda < \lambda_J$, then
- $\omega_k^2 \approx c_s^2 k^2$: has real roots
- giving $\omega_k = c_s k$

so amplitude evolves with linear growth factor

$$D_{k<k_J}(t) \approx \cos(\omega_k t) = \cos\left(\frac{2\pi c_s t}{\lambda}\right) \tag{19}$$

were we took the real part of the complex exponential

physically: **oscillations** about hydrostatic equilib

¬ *Q: what if $k < k_J$, i.e., $\lambda > \lambda_J$?*

Long Wavelength Modes

if $k < k_J$ so $\lambda > \lambda_J$, then
- $\omega_k^2 = -c_s^2 k_J^2$: has imaginary roots
- giving $\omega_k = \pm i c_s k_J$

so linear growth factor is

$$D_{k>k_J}(t) \approx e^{+\omega_k t} \sim e^{t/\tau_{\rm ff}} \tag{20}$$

exponential increase!
$\delta(t) \sim e^{\omega t} \sim e^{+t/t_{\rm freefall}}$
(also an uninteresting decaying mode $e^{-\omega_k t}$)

physically: runaway perturbation growth
∞ **gravitational or Jeans instability**
leading collapse to *gravitational collapse*

What Just Happened?

perturbation evolution sets characteristic physical scales

$$\partial_t^2 \delta - c_s^2 \nabla^2 \delta = 4\pi G \rho_0 \delta$$

- fixed **sound speed** $c_s^2 = \partial P / \partial \rho$
  sets "pressure response speed"
- fixed **freefall time** for unperturbed medium $\tau_{\rm ff}^2 = 1/4\pi G \rho_0$
- from these can form a characteristic distance:
  Jeans length $\lambda_{\rm J} \sim c_s \tau_{\rm ff}$
- also mode-dependent "**crossing time**" $\tau_{\rm crospsing} = \lambda / c_s = 2/pi/\omega_k$
  time for sound wave to cross perturbation of size λ

Perturbation fate:

- if $\lambda < \lambda_J$ then $\tau_{\rm crossing} \ll \tau_{\rm ff}$
  pressure forces have time "organize repsonse" to perturbation
  and exert restoring force: oscillations result!
- but if $\lambda > \lambda_J$ then no time to "organize" restoring force
  collapse ensues!

Intermission: Questions?

Linear Theory I: Newtonian Analysis in Expanding U.

Recall: Newtonian analysis legal for small scales, weak gravity
$\rightarrow$ okay for linear analysis inside Hubble length
apply to **matter-dominated U.**

Coordinate choices

Eulerian time-indep grid $\vec{x}$ fixed in physical space
expansion moves unperturbed fluid elts past as $\vec{x}(t) = a(t)\vec{r}$

Lagrangian coords $\vec{r}$ time-indep but expand in physical space
following fluid element: *locally* comoving

$\Rightarrow$ spatial gradients: $\nabla_{\vec{x}} = (1/a)\nabla_{\vec{r}}$

Now add perturbations $\rho_1 = \rho_0 \delta$, $\vec{v}_1$, Φ_1 simplest to use comoving (Lagrangian) coords

follow observers in unperturbed Hubble flow

perturbation fluid elements $\vec{x}(t) = a(t)\vec{r}(t)$

peculiar fluid velocity $\vec{v}_1(t) = a(t)\vec{u}(t)$

plug in, keep only terms linear in perturbations ($\nabla = \nabla_{\vec{r}}$)

$\rightarrow$ *perturbation evolution to first (leading, linear) order*

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \qquad (21)$$

$$\dot{\delta} = -\nabla \cdot \vec{u} \qquad (22)$$

consider the case of $\Phi_1 = 0$ and $\delta p = 0$, but initial $\vec{u} \neq 0$

Q: what does this represent physically? what happens? why?

Q: implications for the situation when $\Phi_1 \neq 0$ and $\delta\rho \neq 0$?

Velocity Perturbation Evolution

peculiar velocity $\vec{v_1} = a(t)\ \vec{u}$ evolves as

$$\dot{\vec{u}} + 2\frac{\dot{a}}{a}\vec{u} = -\frac{1}{a^2}\nabla\Phi_1 - \frac{1}{a}\frac{\nabla\delta p}{\rho_0} \tag{23}$$

if no pressure nor density perturbations
then $\dot{u} = -2Hu$, and so $u \propto 1/a^2$
and physical speed evolves as $v_1 \propto 1/a$

but recall: long ago derived FLRW test particle speed
evolves as $\vec{v}(t) = \vec{v}_0/a(t)$
$\rightarrow$ pressureless fluid's peculiar vel redshifts same as free particle
$\rightarrow$ expansion acts as "drag" on particles

if $\Phi_1, \delta p \neq 0$: Hubble "drag" still present
removes kinetic energy from collapsing objects
allows total energy to change (decrease) with time
$\rightarrow$ *binding increases!*

Linearized Density Evolution

now look for plane-wave solutions $\leftrightarrow$ write as Fourier modes e.g., $\delta(\vec{r}) \sim e^{-i\vec{k}\cdot\vec{r}}$, with $\vec{k}$ comoving wavenumber

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = \left(4\pi G\rho_0 - \frac{c_s^2 k^2}{a^2}\right)\delta_k \tag{24}$$

if no expansion ($a = 1, \dot{a} = 0$), recover Jeans solution

with expansion:

- Hubble "friction" or "drag" $-2H\dot{\delta}$ opposes density growth
- still critical Jeans scale: $k_J = \sqrt{4\pi G\rho_0 a^2/c_s^2}$
  expect oscillations on small scales, collapse on larger

Unstable Modes: Matter-Dominated U

Consider **large scales** $\lambda \gg \lambda_J$

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k \approx 4\pi G \rho_0 \delta_k \tag{25}$$

in *Matter-dominated U:* $8\pi G\rho/3 = H^2 = (2/3t)^{-2} = 4/9t^2$, so

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{26}$$

Q: how many independent solutions? how to solve?

Matter-dominated U, large scales:

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0 \tag{27}$$

eq homogeneous in $t \to$ try *power law solution*

trial $\delta \sim t^s$ works if

$$s(s-1) + 4s/3 - 2/3 = 0 \tag{28}$$

solutions $s = 2/3, -1$:
growing and decaying modes

$$\delta_+(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3} \ ; \quad \delta_-(t) = \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1} \tag{29}$$

- growing mode dominates
- Hubble friction: exponential collapse softened to power law

★ Note: solutions indep of k *Q: why a big deal?*

Linear Growth Factor

each unstable Fourier mode grows with time as

$$\delta_k(t) \propto D(t) \sim t^{2/3} \sim a \sim \eta^2_{\rm conform} \tag{30}$$

growth independent of wavenumber k

- in k-space, all unstable modes grow by same factor and transform to real space, find
- on large scales (but still subhorizon)

$$\delta(t, \vec{x}_{\rm large}) \simeq D(t)\delta(t_i, \vec{x}_{\rm large}) \tag{31}$$

⇒ entire density contrast pattern grows with same amplification:

⇒ **linear grow factor** $D(t)$ applies to whole field

Q: what would this look like for $\delta(x)$?

Applications to CMB: Naïve Inferences

before decoupling: pressure dominated by photons
$\rightarrow$ expect oscillations – and see them!
after decoupling: growing mode

CMB anisotropies are a snapshot
of perturbations at last scattering
can quantify level: $(\delta T/T)_{\rm ls} \sim 10^{-5}$ at $z_{\rm ls} \sim 1100$

But matter has $\rho \propto a^{-3} \propto T^3$, so $\delta\rho/\rho = 3\delta T/T$
$\rightarrow \delta_{\rm obs}(z = 1100) \sim 3 \times 10^{-5}$ at last scattering
So today, expect fluctuations of size

$$\delta_0 = \frac{D_0}{D_{\rm ls}}\delta_{\rm ls} = \frac{a_0}{a_{\rm ls}}\delta_{\rm ls} = (1 + z_{\rm ls})\delta_{\rm ls} \sim 0.05 \ll 1 \tag{32}$$

Should still be very small–no nonlinear structures, such as us!
Q: obviously wrong–egregiously naïve! What's the flaw?
What's the fix?

Perturbation Growth: Dark Matter vs Baryons

dark matter: pressureless
$\rightarrow$ all k modes unstable if inside Hubble length
but: perturbations grow verry sloooowly during radiation era
$\rightarrow$ DM structures begin formation at matter-radiation equality
then $\delta_{\mathrm{m}}(t) = \delta_{\mathrm{m,init}}\ D(t)$ with $D(t) \propto a(t) \propto t^{2/3}$

baryons: until recomb, tightly coupled to photons
$\rightarrow$ feel huge photon pressure $P_\gamma \propto T^4$
$\rightarrow$ sound speed $c_s \sim c/\sqrt{3}$ huge!
so all sub-horizon modes stable! just oscillate
$\rightarrow$ relativistic pressure-mediated (i.e., acoustic) standing waves!
oscillation frequency $\nu = c_s/\lambda$:
small-scale modes oscillate many times
largest-scale modes $\lambda = c_s \eta_{\mathrm{hor}}$ oscillates only once

Pre-Recombination: Acoustic Oscillations

Baryons in DM-dominated background

$$\ddot{\delta}_b + 2\frac{\dot{a}}{a}\dot{\delta}_b \simeq 4\pi G\rho\delta_{dm} - \frac{k^2 c_s^2}{a^2}\delta_b \sim \frac{\delta_{dm}}{t^2} - \frac{k^2 c_s^2}{a^2}\delta_b \qquad (33)$$

key comparison: mode scale $\lambda \sim k^{-1}$
vs **comoving sound horizon** $c_s t/a = d_{s,com}$

for *large scales* $kc_s t/a \ll 1$: *baryons follow DM*
for *small scales* $kc_s t/a \gg 1$: *baryons oscillate*, as

$$\delta_b \sim \frac{1}{\sqrt{ac_s k}} e^{i\int kc_s d\eta} \qquad (34)$$

(PS 6) where $d\eta = dt/a$ is conformal time

Q: for fixed k, what is δ time behavior?
Q: at fixed t, what is δ pattern vs k?
Q: what sets largest λ that oscillates?

baryonic perturbations do not grow, but oscillate:

$$\delta_b \sim \frac{1}{\sqrt{a c_s k}} e^{i \int k c_s d\eta} \tag{35}$$

to simplify, imagine constant c_s, $\delta_b \sim e^{ikc_s\eta}$

at fixed k, sinusoidal oscillations

phase counts number of cycles $N = kc_s\eta/2\pi = c_s\eta/\lambda$

oscillation frequency: $\omega_k \sim kc_s \sim c_s/\lambda \propto 1/\lambda$

at fixed $t \to$ fixed η:

small λ and large $k \to$ rapid oscillations

largest oscillations at scale $\lambda \sim c_s\eta \sim c_st/a$: *sound horizon*

Q: when do oscillations stop? observable signature?

Cosmic Diversity: Evolution of Multiple Components

Thus far: *implicitly assumed a baryons-only universe*: not ours!

Cosmic "fluid" contains many different species
with different densities, interactions
baryons, photons, neutrinos, dark matter, dark energy

Each component i has its own equations of motion, e.g.:

$$\ddot{\delta}_i + 2H\dot{\delta}_i = -\frac{c_{s,i}^2 k^2}{a^2}\delta_i + 4\pi G\rho_0 \sum_j \Omega_j \delta_j \quad (36)$$

species interact via pressure, gravity: evolution eqs *coupled*
▷ gravity from dominant Ω drives the other components
▷ each species' (pressure) response depends on
microphysics of its interactions, encoded in sound speed $c_{s,i}$

Matter Instability in the Radiation Era

(dark) matter perturbation δ_m during radiation domination

- pick subhorizon scale: growth possible
- focus on $k < k_J$: Jeans unstable (can ignore pressure) and high-k modes just oscillate anyway
- treat radiation perturbations as *smooth*: $\delta_{\rm rad} \approx 0$ $P_r = \rho_r/3$: huge, fast $c_s \sim c$ any perturbations will be oscillatory anyway
- dark matter: weak interactions $\rightarrow$ pressureless $\rightarrow$ $c_s = 0$!

Evolution simple – to rough approximation, for these k:

$$\ddot{\delta}_m + 2\frac{\dot{a}}{a}\dot{\delta}_m \stackrel{\rm rad-dom}{=} \ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m \approx 0 \qquad (37)$$

Simple solutions: growing mode plus decaying mode

$$\delta_m(t) = D(t)\delta_m(t_i) = \left(D_1 \log t + \frac{D_2}{t}\right)\delta_m(t_i) \qquad (38)$$

Q: implications? what about baryons?

Found $D(t) \sim D_1 \log t$: "growing" mode hardly grows!

★ dark matter perturbations *frozen* during rad dom
dark matter growth quenched by
→ non-growth of radiation perturbations
→ extra expansion due to radiation

★ *dark matter perturbation growth stalled*
until end of radiation era: **matter-radiation equality**
i.e., $\rho_{\rm matter} = \rho_{\rm radiation}$ when $z_{\rm eq} \sim 3 \times 10^4$
Q: is before or after BBN? recomb?
⇒ this marks onset of structure formation

Q: how does this update our naive CMB calculation?
Hint: then, correct reasoning for $\delta = \delta_b$ only

baryons tightly coupled to photons till recombination
$\rightarrow$ so dark matter perturbations begin growth earlier

And so: DM has grown more! update earlier estimate and focus on dark matter

$$\delta_{m,0} = \frac{D_{\rm ls}}{D_{\rm eq}} \delta_{b,0} \sim \frac{1+z_{\rm eq}}{1+z_{\rm ls}} \delta_b \sim 30 \times 0.05 \sim 1 \qquad (39)$$

DM can grow to nonlinearity today!

⋆ existence of collapsed cosmic structures
requires collisionless dark matter!
⋆ independent argument for large amounts of
weakly interacting matter throughout universe!

Director's Cut Extras

Unperturbed (zeroth order) eqs,

using $\rho_0 = \rho_0(t)$, $\vec{v}_0 = \frac{\dot{a}}{a}\vec{x} = \dot{a}\vec{r}$

$$\partial_t \rho_0 + \nabla \cdot (\rho_0 \vec{v}) = \dot{\rho_0} + \rho_0 \frac{\dot{a}}{a} \nabla_{\vec{x}} \cdot \vec{x} = 0 \quad (40)$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0 \qquad \Rightarrow \rho_0 \propto a^{-3} \quad (41)$$

Poisson:

$$\nabla^2\Phi_0 = \frac{1}{x^2}\partial_x(x\partial_x\Phi_0) = 4\pi G\rho_0 \;\;\Rightarrow\;\; \Phi_0 = \frac{2\pi G\rho_0}{3}x^2 = \frac{2\pi G\rho_0}{3}a^2r^2$$

$$\nabla_{\vec{x}}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{x} \qquad \nabla_{\vec{r}}\Phi_0 = \frac{4\pi G\rho_0}{3}a\vec{r}$$

Euler

$$d(\dot{a}\vec{r})/dt = \ddot{a}\vec{r} = \frac{\ddot{a}}{a}\vec{x} \;=\; -\frac{4\pi G\rho_0}{3}\vec{x} \qquad (42)$$

$$\frac{\ddot{a}}{a} \;=\; -\frac{4\pi G\rho_0}{3} \qquad (43)$$

Fried accel; with continuity $\rightarrow$ Friedmann

Zeroth order fluid equations $\rightarrow$ usual expanding U
in non-rel approximation

Non-relativistic Cosmic Kinematics

gas particles have random thermal speeds, momenta
how are these affected by cosmic expansion?

Classical picture:
consider non-rel free* particle moving w.r.t. comoving frame
$\vec{\ell}_{\rm com}(t) \neq const$, and so $\vec{\ell}_{\rm phys} = a(t)\ell_{\rm com}(t)$:

$$\begin{aligned} \vec{v} = d\vec{\ell}_{\rm phys}/dt &= \dot{a}(t)\ell_{\rm com}(t) + a(t)\dot{\ell}_{\rm com}(t) \\ &= H\vec{\ell}_{\rm phys} + \vec{v}_{\rm pec} \\ &= \text{Hubble flow} + \text{peculiar velocity} \end{aligned}$$

Note that peculiar velocity v is always w.r.t. the comoving frame–i.e., the particle speed compared to that of a stationary fundamental observer *at the same point*

*i.e., except for gravitation

consider a comoving observer at the origin, $\vec{x} = 0$
in time δt, a particle moves w.r.t. comov frame
physical dist $\delta \vec{x}_{\text{phys}} = \vec{v}_{\text{pec}} \delta t$

but due to Hubble flow, a comoving (fundamental) observer at $\delta \vec{x}_{\text{phys}}$ is moving away from the origin at speed $\vec{v}_{\text{com}} = H \delta \vec{x}_{\text{phys}}$

thus the new speed of the particle relative to its new comoving neighbor is given by the relative velocity
$\vec{v}'_{\text{pec}} = \vec{v}_{\text{pec}} - \vec{v}_{\text{com}}$
(where we used the non-rel velocity addition law)
and so the peculiar velocity *changes* by

$$\delta \vec{v}_{\text{pec}} = -H \delta \vec{x}_{\text{phys}} = -\frac{\dot{a}}{a} \vec{v}_{\text{pec}} \delta t = -\frac{\delta a}{a} \vec{v}_{\text{pec}} \quad (44)$$

Q: physical implications?

$\delta v_{\rm pec}/v_{\rm pec} = -\delta a/a \Rightarrow$ physical peculiar velocity $v_{\rm pec} \propto 1/a$:

- $m v_{\rm non-rel} = p_{\rm non-rel} = p_0/a$
- comoving peculiar velocity $d\ell_{\rm com}/dt \propto 1/a^2$
  slowdown w.r.t. comoving frame: velocity "decays"
  not a "cosmic drag" but rather kinematic effect
  due to struggle to overtake receding of cosmic milestones

Quantum picture:

recall for photons, $p_{\rm rel} = h/\lambda \sim 1/a$ (de Broglie)
but de Broglie holds for matter too: $p_{\rm non-rel} = h/\lambda_{deB} \sim 1/a$
$\Rightarrow$ again, $p_{\rm non-rel} = p_0/a$

true in general, now apply to thermal gas non-relativistic gas: Maxwell-Boltzmann

$$n = \frac{g}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} a^{-3} \int d^3p_0 \; e^{-p_0^2/2mka^2T}$$

if occupation number constant (particle conservation)
need $a^2\,T(a) = T_0 = const$ and thus $T_{\rm non-rel} \propto 1/a^2$:

$$T_{\rm non-rel,decoupled} = \left(\frac{a_{\rm dec}}{a}\right)^2 T_{\rm decoupling} = \left(\frac{1+z}{1+z_{\rm dec}}\right)^2 T_{\rm decoupling}$$

evaluate for $z_{\rm dec} = z_{\rm ri}$: estimate

$$T_{\rm gas,today} \sim \frac{T_{\gamma,0}}{1+z_{\rm dec,gas}} \sim 6 \times 10^{-3}\ {\rm K} \qquad (45)$$

Q: do the experiment...?

Q: what went wrong?

Inhomogeneities: The Spice of Life

So far: we have assumed perfect homogeneity!
If universe strictly homogeneous
indeed would cool to $T_{\mathrm{gas}} \ll T_0$

But happily, U. definitely inhomogeneous on small scales!
gravity amplifies density contrast *Q: why?*
"the rich get richer, the poor get poorer"

this allows for motion, condensation of matter
halo formation, mergers, shocks, star formation, quasars, ...
these overdense structures release energy
lead to diversity of cosmic matter and radiation today!

But how did we get the inhomogeneities?
And what set the primordial composition of baryons?
$\rightarrow$ events in the very early Universe...

Momentum Redshifting: Rigorously

the preceding heuristic arguments give the right result, but to obtain this rigorously requires General Relativity (if you haven't had GR yet, never mind)

in GR: a free particle's motion is a geodesic
so 4-momentum $p^\mu = m dx^\mu/ds = m(\gamma, \gamma\vec{v}) = (E, \vec{p})$ changes as

$$p^\alpha \nabla_\alpha p^\mu = p^\alpha \partial_\alpha p^\mu + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0 \quad (46)$$

and we see that the change in u is due to the connection term Γ, i.e., to curvature
$\rightarrow$ curvature tells matter how to move

note: homogeneity hugely simplifies: $p^\mu = p^\mu(t)$
so $\partial_\mu p = 0$ except for $\partial_t p = \dot{p}$

consider the $\mu = i \in (x,y,z)$ component of the geodesic eq

$$
\begin{aligned}
p^\alpha \partial_\alpha p^i + \Gamma^i_{\alpha\beta} p^\alpha p^\beta &= E\dot{p} + \Gamma^i_{\alpha\beta} p^\alpha p^\beta && (47)\\
&= 0 && (48)
\end{aligned}
$$

note that in FRW, if we write $ds^2 = dt^2 - h_{ij} dx^i dx^j$ where h_{ij} is the spatial metric, then nonzero $\Gamma^i_{\alpha\beta}$ are

$$
\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j \tag{49}
$$

where δ^i_j is the Kronecker delta (try it!)

We then have

$$
E\dot{p}^i + \frac{\dot{a}}{a} E p^i = 0 \tag{50}
$$

and thus

$$
\begin{aligned}
d\vec{p}/dt &= -\frac{\dot{a}}{a}\vec{p} && (51)\\
|\vec{p}| &\propto \frac{1}{a} && (52)
\end{aligned}
$$

Note that this result is completely general, i.e., works for all relativistic p, so

- in non-rel limit, $v \propto 1/a$: vel redshifts, and free particles eventually come to rest wrt the comoving background
- in ultra-rel limit, $v = p/E \approx c$, doesn't redshift, but since $E \approx p$, $E \propto 1/a$: energy redshifts

note classical derivation: didn't need Planck/de Broglie relation $p \propto 1/\lambda$ to show this (though that still works too)

Linear Theory II: Sketch of Relativistic Treatment

see, e.g., Dodelson text, Liddle & Lyth Ch. 14

Recall limits of Newtonian treatment:

- only appropriate for scales $\lambda \ll d_H$: sub-horizon
- relativistic effects like time dilation absent or *ad hoc*

General Relativistic approach to cosmological perturbations

- as in Newtonian analysis, perturb density, velocity
  $\rightarrow$ this perturbs stress-energy
  schematically "$\delta T \approx \delta\rho + \delta P = \delta\rho + c_s^2 \delta\rho$"
- must therefore add small perturbations to metric:
  $g_{\mu\nu} = g_{\mu\nu}^{\mathsf{FRW}} + h_{\mu\nu}$
- these are related by Einstein's Equation
  $G_{\mu\nu} \approx$ "$\partial\partial g^{\mathsf{FRW}} + \partial\partial h$" $= 8\pi G_N T_{\mu\nu} \approx$ "$8\pi G_N(\rho + \delta\rho)$"

Metric Perturbations

Perturbations to metric tensor can be classified as:

- *scalar* – density perturbations couple to these
  these are most important
- *vector* – velocity perturbations couple to these
  these are least important (perturbations decay with time)
- *tensor* – source of gravity waves
  inflationary quantum perturbation excite these modes!

focus on *scalar* perturbations, which modify FRW metric thusly:

$$(ds^2)_{\text{perturbed}} = a(\eta)^2 \left[(1+2\Psi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j\right] \quad (53)$$

Coordinate freedom $\leftrightarrow$ "gauge" choice $\leftrightarrow$ spacetime "slicing"

$\Rightarrow$ here: *"conformal Newtonian gauge"*:

- $\Psi(\vec{x},t), \Phi(\vec{x},t)$ Schwarzchild-like forms if $a=1, \dot{a}=0$

Substitute perturbed metric into Einstein, keep only linear terms in Φ and Ψ, e.g., neglect Φ^2

use conformal time

and go to k-space

- $\nabla_\mu T^{\mu 0} \to$ "continuity"

$$\frac{d\delta}{d\eta} + ikv + 3\frac{d\Phi}{d\eta} = 0 \tag{54}$$

- $\nabla_\mu T^{\mu i} \to$ "Euler"

$$\frac{dv}{d\eta} + \frac{da/d\eta}{a} v + ik\Psi = \text{pressure sources} \tag{55}$$

- $G_{\mu\nu} = 8\pi G_\mathrm{N} T_{\mu\nu} \to$ "Poisson"

$$k^2\Phi = -4\pi G a^2 \rho\delta \tag{56}$$

$$k^2(\Psi - \Phi) = -8\pi G a^2 \text{“}\langle P_x - P_y\rangle\text{”} \tag{57}$$

expect *anisotropic stress* small: $\langle P_x - P_y\rangle \ll \rho\delta \to$ $\boxed{\Psi \approx \Phi}$

Recall: conformal time η gives particle horizon

On *sub-horizon* scales $\lambda \sim 1/k \ll \eta$:
relativistic treatment gives back Newtonian result
in fact: justifies our Newtonian treatment

On *super-horizon* scales $\lambda \sim 1/k \gg \eta$:
relativistic treatment still valid
$\rightarrow$ will use this to follow inflationary perturbations
through horizon crossing