Astro 507

Lecture 9

Feb. 10, 2020

Announcements:

- **Preflight 2 posted, due noon Friday**
  includes discussion question on the Anthropic Principle!

Last time: the mass-energy budget of the cosmos

*Q: why do we want to know the total mass-energy budget today?*

*Q: two what do we compare this?*

*Q: result using pure theory? using galaxies?*

# Density and Destiny

Enough generalities! What about *our* real Universe?
Fate (and geometry) of U. depend on
current values of $\Omega_{i,0} = \rho_{i,0}/\rho_{\rm crit,0}$
and $\Omega_0 = \sum \Omega_i$ where

$$
\begin{aligned}
\rho_{\rm crit,0} &= \frac{3H_0^2}{8\pi G} \\
&= 1.9 \times 10^{-29}\ h^2\ {\rm g/cm^{-3}} \approx 10^{-29}\ {\rm g/cm^{-3}} \\
&= 2.78 \times 10^{11}\ h^2\ M_\odot\ {\rm Mpc^{-3}} \approx 1.4 \times 10^{11}\ M_\odot\ {\rm Mpc^{-3}} \\
&\approx 6\ {\rm H\,atoms\ m^{-3}}
\end{aligned}
$$

**Methods Of Estimating Cosmic Density**

⋆ **Pure Theory** $\Omega = 1$

⋆ **Galaxies** $\Omega_{\rm stars+gas} \simeq 0.0024 \pm 0.0012$, $\Omega_{\rm lum+halo} \simeq 0.02$

*Q: implications? what if this is a fair sample?*

*Q: why would/wouldn't it be?*

**cosmic mass/light sample: galaxy clusters**

can find cluster $M_{\rm tot}$ from several methods

e.g., `www: cluster gravitational lens`

$\Upsilon_{\rm cluster} \sim 300h \rightarrow \Omega_{\rm cluster} \sim 0.25h^{-1} \sim 0.3$

Note: since $\Upsilon_{\rm cluster} > \Upsilon_{\rm halos}$

$\rightarrow$ immediately conclude that *halos are not fair sample*

$\rightarrow$ i.e., halos miss extra dark matter on larger scales

$\rightarrow$ hints for galaxy formation...

...but clusters have $\delta\rho/\rho_0 \sim 1$

$\rightarrow$ largest bound objects

$\rightarrow$ should be fair sample:

$\Rightarrow \Omega_{\rm matter} \sim 0.3$ (including DM!)

Cosmic Density Measurement Procedure II:
Microwave background anisotropies

CMB temperature anisotropies sensitive to cosmic geometry
`www: Planck 2013 results + other observations (BAO)`

$$\Omega_\kappa \equiv 1 - \Omega_0 = 0.0005 \pm 0.0033$$
$$\boxed{\Omega_0 = 1.0005 \pm 0.0033!}$$

$\Rightarrow$ $\Omega_0 = 1$ to $\sim 0.3\%$ level!!!
$\Rightarrow$ *a flat universe! theory prejudice correct!*

but: $\Omega_{\rm matter} \approx 0.27$ (including DM!)
$\rightarrow$ $\Omega_{\rm other} = 0.73$?!?



*Who ordered that? What is the other, dominant component?*
$\Lambda$? "dark energy" ?!?

# Beyond Newton

Thus far: Newtonian cosmology
- develops intuition
- correct over small scales $\ll d_H$

Shortcomings:
- some features "pulled of out a hat"
  e.g., curvature scale $R$
  presence, coefficient of pressure
- Newtonian physics is incomplete (=wrong!)

$\Rightarrow$ *the Universe is relativistic!*

# General Relativity

# Relativity for the Impatient Cosmologist

For *General Relativity newcomers*, we will:

- sketch how GR generalizes special relativity
- sketch basic concepts of GR
- qualitatively discuss similarities, differences with special relativity, Newtonian Gravity
- No substitute for a real, rigorous, in-depth course: *take General Relativity!*

For *General Relativity veterans*, we will:

- sketch how Einstein equations $\rightarrow$ Friedmann eqs

For *everyone*, we will:

- show how the Cosmological Principle strongly constrains possible cosmic spacetimes
- semi-derive the cosmic (FLRW) metric
- use this to probe lifestyles in an expanding universe

# Spacetime

see S. Carroll, *Spacetime and Geometry*; R. Geroch, *General Relativity from A to B*

evolving view of space, time, and motion:
Aristotle $\rightarrow$ Galileo $\rightarrow$ Einstein

Key basic concept: **event**
*occurrence localized in space and time*
e.g., firecracker, finger snap
idealized $\rightarrow$ no spatial extent, no duration in time

a goal (*the* goal?) of physics:
describe relationships among events

∞

*Q: consider collection of all possible events–what's included?*

# Spacetime Coordinates

Each event specifies a unique point in
spacetime = collection of all events

lay down coordinate system: 3 space coords, one time
4-dimensional, but as yet time & space always "orthogonal"

example:
a time $t$ and Cartesian $(x, y, z)$
specify event $\rightarrow$ $(t, x, y, z)$

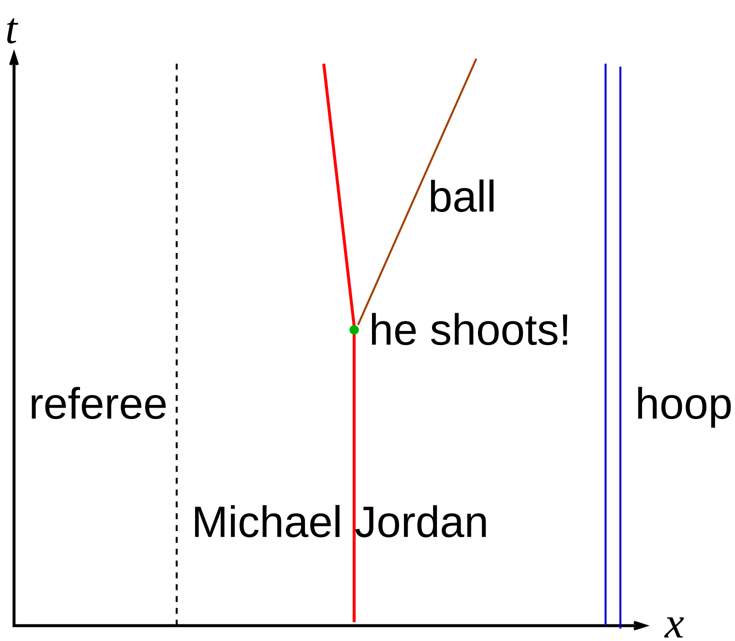
physics asks (and answers) what is the relationship
between two events, e.g., $(t_1, x_1, y_1, z_1)$ and $(t_2, x_2, y_2, z_2)$

Represent spacetime geometrically: *spacetime diagram*
e.g., plot $(x, t)$ coordinate plane
*Q: one event? one observer at rest? a jump shot?*

# Spacetime Diagram

objects (observers) at *rest*:
same $x, y, z$ always, $t$ ticks forward
geometrically, a line in spacetime: **“world line”**
*if at rest: world line vertical*

*constant speed:* $x = vt$*: diagonal line*



# Galilean Relativity

consider two identical laboratories
(same apparatus, scientists, funding, vending machines)
move at constant velocity wrt each other

Galileo:
no experiment either can do (without looking outside)
will answer "which lab is moving"
$\rightarrow$ *no absolute motion*, only relative velocity

Newton: laws of mechanics invariant
for observers moving at const v
"inertial observers"

Implications for spacetime
no absolute motion $\rightarrow$ *no absolute space*
(but still no reason to abandon absolute time)

# Trouble for Galileo

Maxwell: equations govern light
very successful, but:

- predicts unique (constant) light speed $c$–relative to whom?
- Maxwell eqs **not** Galilean invariant

Lorentz: Maxwell eqs invariant when

$$t' = \gamma(t - vx/c^2) \tag{1}$$
$$x' = \gamma(x - vt) \tag{2}$$
$$y' = y \tag{3}$$
$$z' = z \tag{4}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2} \geq 1$

Einstein:
Lorentz transformation not just a trick
but correct relationship between inertial frames!
$\Rightarrow$ this is the way the world is

# Einstein: Special Relativity

consider two events
$(t, x, y, z)$ and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$

different inertial observers *disagree* about
i.e., measure different values for: $\Delta t$ *and* $\Delta \vec{r}$

but all *agree* on = calculate same value of
the **interval**

$$\Delta s^2 \equiv (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (5)$$
$$= (c\Delta t)^2 - (\Delta \ell)^2 \quad (6)$$

everyone agrees on value = *Lorentz invariant*
Note: interval can have $\Delta s^2 > 0, < 0, = 0$

*Light pulse:*

in rest frame of flash: photon positions $\Delta\ell = c\Delta t$

calculate interval: $\Delta s_{\text{light}} = 0$

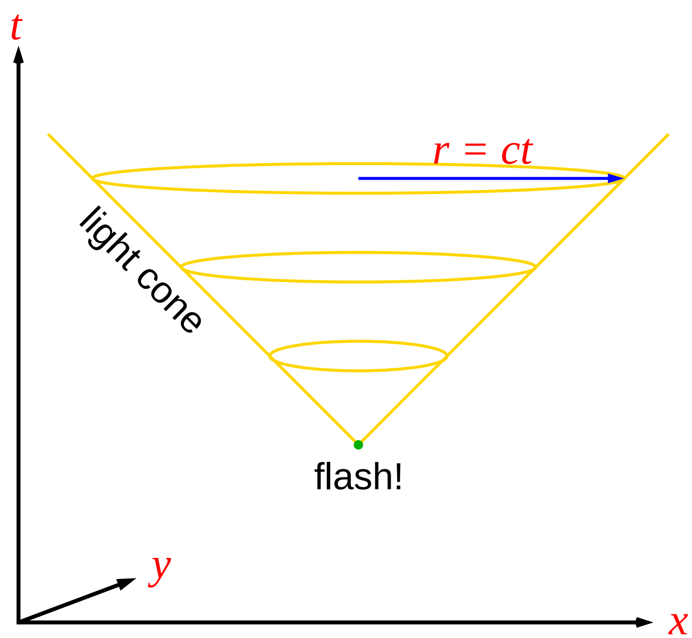
$\rightarrow$ *light moves at $c$ in all frames!*

all observers measure same speed of light!

*Q: light flash in spacetime diag?*

# Light Pulse in Spacetime

in spacetime: light pulse at origin $(t, x, y, z) = (0, 0, 0, 0)$
moves so that distance $r = \sqrt{x^2 + y^2 + z^2} = ct$
geometrically: **light cone**



Motion and time:

Consider *two events, at rest in one frame:*

$\Delta\vec{x}_{\text{rest}} = 0$ in rest frame, so

$\Delta s = c\Delta t_{\text{rest}}$: $c\times$ elapsed time in rest frame

*In another inertial frame*, relative speed $v$:

events separated in space by $\Delta x' = v\Delta t'$

$$\Delta s = \sqrt{c^2 \Delta t'^2 - \Delta x'^2} = \sqrt{c^2 - v^2}\ \Delta t' = \frac{1}{\gamma} c\Delta t' \tag{7}$$

since $\Delta s$ same: infer $\Delta t' = \gamma \Delta t_{\text{rest}} > \Delta t_{\text{rest}}$

$\Rightarrow$ moving clocks appear to run slow

(special) relativistic time dilation

$\Rightarrow$ no absolute time (and no absolute space)



Note: more on Special Relativity in Director's Cut Extras to today's notes

H. Minkowski:

"Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

# The Speed of Massive Particles

Special relativity general rule: $v = p/E$
where $E$ is total energy (see Extras to notes)
good for particles of any mass $m \geq 0$
...and where we have and will set $c = 1$
you can show that with explicit $c$ factors, $v/c = cp/E$

but $E$ and $p$ also connected via
*invariant* $E^2 - p^2 = m^2$

$$v = \frac{\sqrt{E^2 - m^2}}{E} = \sqrt{1 - \left(\frac{m}{E}\right)^2} \tag{8}$$

*Q: implications? what if $m = 0$? $m \neq 0$?*

## Director's Cut Extras: Special Relativity

# Pre-Relativity: Aristotle

$x, y, z$ Cartesian (Euclidean geometry)
spatial distance $\ell$ between events is:

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{9}$$

and is independent of time
elapsed time between events is: $t_2 - t_1$
and is independent of space
"absolute space" and "absolute time"

Is a particle at rest? $\Leftrightarrow$ do $(x, y, z)$ change?
$\rightarrow$ "absolute rest, absolute motion"



*Diagram: Aristotelian spacetime*
unique "stacking" of "time slices"

# Life According to Aristotle

Note: even in absolute space
event location indep of coordinate description
e.g., two observers choose coordinates different by a rotation:
$(x, y)$ and $(x', y') = (x \cos\theta - y \sin\theta, y \cos\theta + x \sin\theta)$
preserves distance from origin: $x^2 + y^2 = (x')^2 + (y')^2$

objects (observers) at rest:
same $x, y, z$ always, $t$ ticks forward
geometrically, a line in spacetime: **"world line"**
if at rest: world line vertical
constant speed: $x = vt$: diagonal line

light: moves at "speed of light" $c$
$\rightarrow$ well-defined, since motion absolute
in spacetime: light pulse at origin $(t, x, y, z) = (0, 0, 0, 0)$
moves so that distance $\ell = \sqrt{x^2 + y^2 + z^2} = ct$
geometrically: **light cone**

# Galilean Frames

each inertial obs has own personal frame:
obs ("Angelina") at rest in own frame: $(x, y, z)$ same for all $t$
but to another obs ("Brad") in relative motion $\vec{v} = v\hat{x}$
B sees A's frame as time-dependent:

$$x_{\text{A as seen by B}} = x' \;=\; x - vt \tag{10}$$

but still absolute time: $t' = t$
Newton's laws (and Gravity) hold in both frames
can show: $d^2\vec{r}/dt^2 = \vec{F}(\vec{r}) \;\Rightarrow\; d^2\vec{r}'/dt'^2 = \vec{F}(\vec{r}')$
"Galilean invariance"

Geometrically:
different inertial frames $\rightarrow$ transformation of spacetime
slide the "space slices" at each time
(picture "shear," or beveling a deck of cards)

# Spacetime and Relativity

Pre-Relativity: space and time separate and independent
but *rotations* mix *space* coords, e.g.,

$$x' = x\cos\theta - y\sin\theta \quad ; \quad y' = y\cos\theta + x\sin\theta \qquad (11)$$

without changing underlying vector (rotation of coords only)
transform rule holds not only for $\vec{x}$
but all other physical directed quantities: e.g., $\vec{v}, \vec{a}, \vec{p}, \vec{g}, \vec{E}$

Lesson: express & guarantee underlying rotational invariance
by writing physical law in vector form
e.g., $\vec{F} = m\vec{a}$ gives same physics for any coord rotation

In special relativity:
spatial rotations still allowed, but also...

"boosts" from one frame to another
with relative speed $\vec{v} = v\hat{x}$

$$t' = \gamma(t - vx/c^2) \quad (12)$$
$$x' = \gamma(x - vt) \quad (13)$$
$$y' = y \quad (14)$$
$$z' = z \quad (15)$$

- truly mix space and time $\rightarrow$ spacetime
- look like rotations, but 4-dimensional

$\rightarrow$ should express laws in terms of 4-D vectors:
"4-vectors," $t, x$ components transform via Lorentz

# Velocity, Momentum, Energy

Velocity:

for events separated by $dx^\mu = (dt, d\vec{x})$, put

$$u^\mu = \frac{dx^\mu}{ds} = (\frac{dt}{ds}, \frac{d\vec{x}}{ds}) \tag{16}$$

covariant: written this way, a 4-vector:

transforms in boost a la Lorentz

i.e., *components are different* in different frames

but underlying physical entity frame-independent

"like with space vectors and rotations"

norm ("length") of 4-velocity

$$u \cdot u = \left(\frac{dt}{ds}\right)^2 - \left(\frac{d\vec{x}}{ds}\right)^2 = \frac{dt^2 - d\vec{x}^2}{ds^2} = \frac{ds^2}{ds^2} = 1$$

same number for all observers: *invariant*

Now want 4-momentum $p^\mu$:
consider particle of (rest) mass $m$
where: rel. $p^i$ should go to $m\vec{v}$ for small $v$
try: $p^\mu = mcu^\mu$
space part: $\boxed{\vec{p} = \gamma m\vec{v}}$ rel momentum
time part:

$$p^0 = \gamma mc \approx \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2\right) = \frac{1}{c}\left(mc^2 + K\right) \tag{17}$$

can identify $\boxed{E_{\text{rel,tot}} = cp^0}$, but then
rest mass has energy $E_{\text{rest}} = mc^2$!

energy, momentum conservation $\rightarrow p^\mu$ cons
compact, unified treatment:
$(p^\mu)_{\text{init}} = (p^\mu)_{\text{fin}}$ (4 equations)

# The Charms of 4-Momentum

Invariant norm (everyone agrees)

$$p \cdot p = (p^0)^2 - (\vec{p})^2 = E^2 - \vec{p}^2 = m^2 \tag{18}$$

- rel. (total) energy is $E(p) = \sqrt{(cp)^2 + (mc^2)^2}$
- in rest frame: $\vec{p} = 0 \to E = mc^2$ "rest mass energy"
- define rel kinetic energy: $K_{\text{rel}} = E - mc^2$
  can show: $K_{\text{rel}} \to p^2/2m$ if $v \ll c$

Velocity
can show: $\vec{p}/E_{\text{tot}} = \vec{v}$
- non-rel: $Q$?

What if $m = 0$?
- $E^2 - \vec{p}^2 = 0 \to E = cp$: E is "all kinetic"
- $v = p/E = 1 = c$: moves at $c$ always!

# World Lines and Dynamics

for any observer (i.e., any coordinate system):
events along own worldline have

$$(\Delta s)^2 = (\text{observer's apparent elapsed time})^2 \tag{19}$$

*Q: why?*

observers' total elapsed time going from events $A \to B$: $\Delta t = \int_a^b ds$
generically: in frame $x'$, elapsed time: $\Delta t = \int_a^b \sqrt{1 - v^2}\, dt'$

consider "race" from event $A$ to event $B$
accelerated vs non-accelerated ("free") observers
*Q: physical picture?*
can show: everyone agrees that
*non-accelerated* observer measures *longest* $\Delta t$
*Q: this is huge–why? what's special about such observers in SR?*

non-accelerated observer $\rightarrow$ no forces
i.e,. a free body!

so in Special Relativity:
of all trajectories from events $A \rightarrow B$
free bodies have max $\int ds$

but free body trajectory is natural motion!

Implications
$\Rightarrow$ free body follows extremum of $\int ds$
law of motion!
i.e., variation $\delta \int ds = 0$ selects physical worldline!
$\Rightarrow$ twin "paradox" is not *Q: why?*